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Cover image by Samia Khalaf, assisted by Jason Challas. Samia, working her way towards a career in art and design, is an animation student at West Valley College in Saratoga, California, where Jason teaches. As noted on page 169, all animal transformations are completely reversible. Page 239 art by Susan Stromquist.

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MATHEMATICS MAGAZINE

LETTER FROM THE EDITOR

The cover refers to Mad Vet puzzles, in which animals are transformed into other animals. These puzzles are the starting point for the article by Gene Abrams and Jessica Sklar in this issue. They show how each of these puzzles is related to a particular semigroup. Understand the semigroup and solve the puzzle! From there they find connections to graph theory and to current research.

Other animals—some horses, but also beasts like Lebesgue measure—take the stage when Julia Barnes and Lorelei Koss invite us to their carnival. It is a carnival of mappings, exploring the implications of G. D. Birkhoff's Ergodic Theorem.

Ever drill a hole through the center of a sphere? In calculus problems, perhaps. Vincent Coll and Jeff Dodd consider what other solids you might drill through instead. The diameters of the Earth and of a hydrogen atom are mentioned.

Danielle Arett and Suzanne Dorée tell us about Tower of Hanoi graphs. They explore properties of these graphs and use them to derive combinatorial identities. Arett was Dorée's student at Augsburg College when this work began.

In the Notes section, Todd Will gives us a definitive treatment of a sums-of-squares problem, partly by combining (and sometimes reconciling) old results. There are also pieces by Ron Hirshon on random walks with barriers (or gambling games, if we prefer), Christopher Frayer on polynomial root squeezing, and Alexander Kheifets and James Propp on integration by parts. At the back of the issue are problems, solutions, and results from the 50th International Mathematical Olympiad.

But let us begin with some beginnings. Ko-Wei Lih introduces us to a magic square from 18th-century Korea—long before Euler's work on the latin squares. Could Choe's square have influenced Benjamin Franklin? He would surely have been interested, and it was in print before he was ten years old.

Walter Stromquist, Editor

ARTICLES

A Remarkable Euler Square before Euler

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Orthogonal Latin squares and Choe's configuration

A *Latin square* of order n is formed when the cells of an $n \times n$ square array are filled with elements taken from a set of cardinality n so that all cells along any row or any column are occupied with distinct elements. A notion of orthogonality between two Latin squares can be defined as follows. We may juxtapose two Latin squares A and B of order n into one square array so that each cell is occupied with an ordered pair, first component from A and second component from B. When all n^2 of these ordered pairs are distinct, we say that A is orthogonal to B. Obviously, this orthogonality relation is symmetric. The juxtaposition of two orthogonal Latin squares is called a *Graeco-Latin square* by Euler, who was the first to study the properties of Latin and Graeco-Latin squares in a short paper [**2**] written in 1776. His motivation was to produce magic squares from Graeco-Latin squares. We call a Graeco-Latin square an *Euler square* in this article.

A *magic square* of order n is an arrangement of the numbers $1, 2, \ldots, n^2$ into an $n \times n$ square array so that the sum of numbers along any row, any column, or either of the two main diagonals is equal to the fixed number $n(n^2 + 1)/2$.

To make things simpler, we always suppose that a Latin square of order n is filled with numbers from the set $\{1, 2, \ldots, n\}$. Euler used the simple algorithm of mapping the pair (x, y) into the number $n(x - 1) + y$ to convert a Graeco-Latin square of order n into an array of order n. We call this mapping the *canonical mapping* in the sequel. It is easy to see that the range of this mapping is the set $\{1, 2, \ldots, n^2\}$ and the sum of numbers along any row or column of the array is $n(n^2 + 1)/2$. If we can arrange to have both main diagonals sum to $n(n^2 + 1)/2$, then a magic square is produced.

The highest order of an Euler square explicitly constructed in [**2**] is five. The following is an example from [**2**] in matrix form with entry xy representing a pair (x, y) in the Euler square. Applying the canonical mapping to this square, we obtain the magic square on the right.

$$\begin{pmatrix} 34 & 45 & 51 & 12 & 23 \\ 25 & 31 & 42 & 53 & 14 \\ 11 & 22 & 33 & 44 & 55 \\ 52 & 13 & 24 & 35 & 41 \\ 43 & 54 & 15 & 21 & 32 \end{pmatrix}$$

14	20	21	2	8
10	11	17	23	4
1	7	13	19	25
22	3	9	15	16
18	24	5	6	12

Math. Mag. **83** (2010) 163–167. doi:10.4169/002557010X494805.

Orthogonal Latin squares have been known to predate Euler in Europe. A comprehensive history of Latin squares can be found in [**1**]. However, it is surprising that an Euler square of order higher than five was already in existence in the Orient, prior to Euler's paper. In a Korean mathematical treatise *Kusuryak* (九數略, Summary of the Nine Branches of Numbers) written by Choe Sŏk-chŏng (崔錫鼎, 1646–1715), an Euler square of order nine appeared. Choe, a Confucian scholar and one time the prime minister of the Choson Dynasty, wrote his treatise presumably after his retirement in 1710. Figure 1 is a facsimile of the pages copied from [**5**] (vol. 1, pp. 698–699) exhibiting Choe's configurations. The 9×9 square on the right is our main concern in this note. (The square begins with the rightmost column on the left-hand page and extends over most of the right-hand page.)

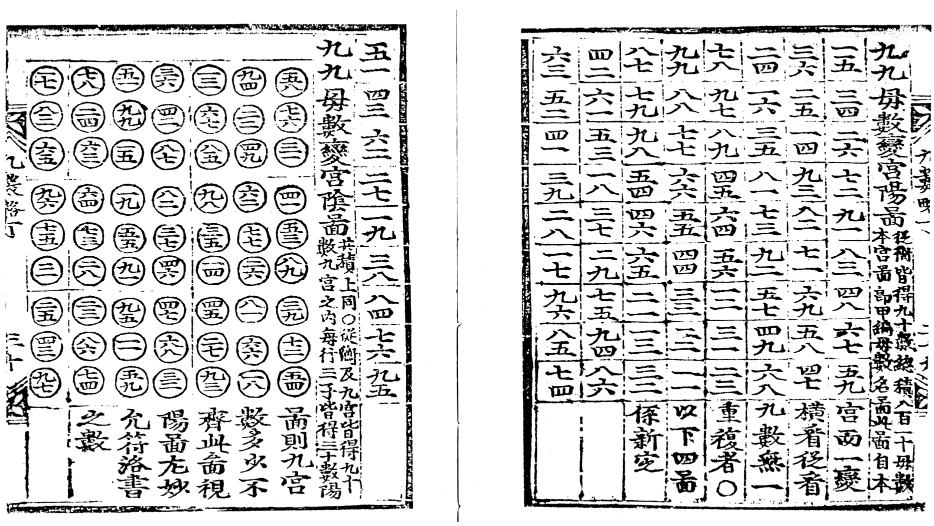

Figure 1 A facsimile of Choe's configurations

The reader is referred to [**3**] and [**4**] for background information on the history of Korean mathematics. Choe's treatise was entirely written in Chinese characters. He did not reveal any clue as how he arrived at his configurations. A modern matrix form M of his square is displayed as follows.

$$M = \begin{pmatrix} 51 & 63 & 42 & 87 & 99 & 78 & 24 & 36 & 15 \\ 43 & 52 & 61 & 79 & 88 & 97 & 16 & 25 & 34 \\ 62 & 41 & 53 & 98 & 77 & 89 & 35 & 14 & 26 \\ 27 & 39 & 18 & 54 & 66 & 45 & 81 & 93 & 72 \\ 19 & 28 & 37 & 46 & 55 & 64 & 73 & 82 & 91 \\ 38 & 17 & 29 & 65 & 44 & 56 & 92 & 71 & 83 \\ 84 & 96 & 75 & 21 & 33 & 12 & 57 & 69 & 48 \\ 76 & 85 & 94 & 13 & 22 & 31 & 49 & 58 & 67 \\ 95 & 74 & 86 & 32 & 11 & 23 & 68 & 47 & 59 \end{pmatrix}$$

Hong-Yeop Song has called attention to this square in [**6**]. As observed in [**6**], the following square is obtained when the canonical mapping is applied to M.

37	48	29	70	81	62	13	24	5
30	38	46	63	71	79	6	14	22
47	28	39	80	61	72	23	4	15
16	27	8	40	51	32	64	75	56
9	17	25	33	41	49	57	65	73
26	7	18	50	31	42	74	55	66
67	78	59	10	21	2	43	54	35
60	68	76	3	11	19	36	44	52
77	58	69	20	1	12	53	34	45

Choe's square M is a juxtaposition of the following two Latin squares L and R. We write $M = L \odot R$, where $\odot$ is a notation for the juxtaposition operation.

$$L = \begin{pmatrix} 5 & 6 & 4 & 8 & 9 & 7 & 2 & 3 & 1 \\ 4 & 5 & 6 & 7 & 8 & 9 & 1 & 2 & 3 \\ 6 & 4 & 5 & 9 & 7 & 8 & 3 & 1 & 2 \\ 2 & 3 & 1 & 5 & 6 & 4 & 8 & 9 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 2 & 6 & 4 & 5 & 9 & 7 & 8 \\ 8 & 9 & 7 & 2 & 3 & 1 & 5 & 6 & 4 \\ 7 & 8 & 9 & 1 & 2 & 3 & 4 & 5 & 6 \\ 9 & 7 & 8 & 3 & 1 & 2 & 6 & 4 & 5 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 3 & 2 & 7 & 9 & 8 & 4 & 6 & 5 \\ 3 & 2 & 1 & 9 & 8 & 7 & 6 & 5 & 4 \\ 2 & 1 & 3 & 8 & 7 & 9 & 5 & 4 & 6 \\ 7 & 9 & 8 & 4 & 6 & 5 & 1 & 3 & 2 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 8 & 7 & 9 & 5 & 4 & 6 & 2 & 1 & 3 \\ 4 & 6 & 5 & 1 & 3 & 2 & 7 & 9 & 8 \\ 6 & 5 & 4 & 3 & 2 & 1 & 9 & 8 & 7 \\ 5 & 4 & 6 & 2 & 1 & 3 & 8 & 7 & 9 \end{pmatrix}$$

It is also observed in [**6**] that each pair of corresponding rows of L and R form a palindrome. Let $P_n = (p_{i,j})$ be an $n \times n$ permutation matrix with $p_{i,j} = 1$ when $j = n + 1 - i$. Then this observation amounts to the matrix equality $R = LP_9$.

In the next section, we list new observations about nice properties of M. In the last section we will explain how M can be constructed by a matrix product method. The construction will make clear why these properties hold.

More nice properties of Choe's square

Sums of centrally symmetric cells Any pair of cells in a matrix of odd order is said to be *centrally symmetric* if they are located symmetrically with respect to the center cell. In the square L (or R), any pair of entries at centrally symmetric cells sum to 10. It follows that, in Choe's square M, if we read each entry as a two-digit integer, any pair of centrally symmetric entries sums to 110. (In the magic square formed by the canonical map, any pair of centrally symmetric entries sums to 82.)

A partition into orthogonal Latin squares We split M right down the central vertical line to get two matrices L' and R', each of which is a Latin square.

$$L' = \begin{pmatrix} 5 & 1 & 6 & 3 & 4 & 2 & 8 & 7 & 9 \\ 4 & 3 & 5 & 2 & 6 & 1 & 7 & 9 & 8 \\ 6 & 2 & 4 & 1 & 5 & 3 & 9 & 8 & 7 \\ 2 & 7 & 3 & 9 & 1 & 8 & 5 & 4 & 6 \\ 1 & 9 & 2 & 8 & 3 & 7 & 4 & 6 & 5 \\ 3 & 8 & 1 & 7 & 2 & 9 & 6 & 5 & 4 \\ 8 & 4 & 9 & 6 & 7 & 5 & 2 & 1 & 3 \\ 7 & 6 & 8 & 5 & 9 & 4 & 1 & 3 & 2 \\ 9 & 5 & 7 & 4 & 8 & 6 & 3 & 2 & 1 \end{pmatrix} \qquad R' = \begin{pmatrix} 9 & 7 & 8 & 2 & 4 & 3 & 6 & 1 & 5 \\ 8 & 9 & 7 & 1 & 6 & 2 & 5 & 3 & 4 \\ 7 & 8 & 9 & 3 & 5 & 1 & 4 & 2 & 6 \\ 6 & 4 & 5 & 8 & 1 & 9 & 3 & 7 & 2 \\ 5 & 6 & 4 & 7 & 3 & 8 & 2 & 9 & 1 \\ 4 & 5 & 6 & 9 & 2 & 7 & 1 & 8 & 3 \\ 3 & 1 & 2 & 5 & 7 & 6 & 9 & 4 & 8 \\ 2 & 3 & 1 & 4 & 9 & 5 & 8 & 6 & 7 \\ 1 & 2 & 3 & 6 & 8 & 4 & 7 & 5 & 9 \end{pmatrix}$$

Again, $R' = L'P_9$ and $L' \odot R'$ is an Euler square.

$$\begin{pmatrix} 59 & 17 & 68 & 32 & 44 & 23 & 86 & 71 & 95 \\ 48 & 39 & 57 & 21 & 66 & 12 & 75 & 93 & 84 \\ 67 & 28 & 49 & 13 & 55 & 31 & 94 & 82 & 76 \\ 26 & 74 & 35 & 98 & 11 & 89 & 53 & 47 & 62 \\ 15 & 96 & 24 & 87 & 33 & 78 & 42 & 69 & 51 \\ 34 & 85 & 16 & 79 & 22 & 97 & 61 & 58 & 43 \\ 83 & 41 & 92 & 65 & 77 & 56 & 29 & 14 & 38 \\ 72 & 63 & 81 & 54 & 99 & 45 & 18 & 36 & 27 \\ 91 & 52 & 73 & 46 & 88 & 64 & 37 & 25 & 19 \end{pmatrix}$$

However, the canonical mapping does not convert $L' \odot R'$ into a magic square.

Exchanges of four pairs of centrally symmetric cells We consider the following four pairs of centrally symmetric cells in the matrix $M = (m_{i,j})$:

$$\{m_{i,i}, m_{10-i,10-i}\}, \{m_{i,5}, m_{10-i,5}\}, \{m_{5,i}, m_{5,10-i}\}, \{m_{i,10-i}, m_{10-i,i}\}.$$

For each i, $1 \le i \le 4$, if we simultaneously interchange the entries in each of the above four pairs, we get an Euler square M_i. Each M_i can be converted into a magic square by the canonical mapping. If we split each M_i along the central vertical line to get two Latin squares L'_i and R'_i, then $R'_i = L'_i P_9$ and $L'_i \odot R'_i$ is again an Euler square.

Our method to construct Choe's square

First we define a *formal Kronecker product* of two matrices. Let $U = (u_{i,j})$ be an $m \times m$ matrix and $V = (v_{i,j})$ be an $n \times n$ matrix. Define $U \otimes V$ to be an $mn \times mn$ matrix

$$\begin{pmatrix} Y_{1,1} & Y_{1,2} & \cdots & Y_{1,m} \\ Y_{2,1} & Y_{2,2} & \cdots & Y_{2,m} \\ \cdots & \cdots & \cdots & \cdots \\ Y_{m,1} & Y_{m,2} & \cdots & Y_{m,m} \end{pmatrix},$$

where $Y_{i,j}$ is an $n \times n$ matrix whose (s,t)-entry is equal to the pair $(u_{i,j}, v_{s,t})$.

There are six permutations of the numbers 1, 2, and 3. They can be grouped into two 3×3 orthogonal Latin squares A and B such that $B = AP_3$.

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

Now $A \otimes A$ is the following matrix.

$$\begin{pmatrix}
(2,2) & (2,3) & (2,1) & (3,2) & (3,3) & (3,1) & (1,2) & (1,3) & (1,1) \\
(2,1) & (2,2) & (2,3) & (3,1) & (3,2) & (3,3) & (1,1) & (1,2) & (1,3) \\
(2,3) & (2,1) & (2,2) & (3,3) & (3,1) & (3,2) & (1,3) & (1,1) & (1,2) \\
(1,2) & (1,3) & (1,1) & (2,2) & (2,3) & (2,1) & (3,2) & (3,3) & (3,1) \\
(1,1) & (1,2) & (1,3) & (2,1) & (2,2) & (2,3) & (3,1) & (3,2) & (3,3) \\
(1,3) & (1,1) & (1,2) & (2,3) & (2,1) & (2,2) & (3,3) & (3,1) & (3,2) \\
(3,2) & (3,3) & (3,1) & (1,2) & (1,3) & (1,1) & (2,2) & (2,3) & (2,1) \\
(3,1) & (3,2) & (3,3) & (1,1) & (1,2) & (1,3) & (2,1) & (2,2) & (2,3) \\
(3,3) & (3,1) & (3,2) & (1,3) & (1,1) & (1,2) & (2,3) & (2,1) & (2,2)
\end{pmatrix}$$

Next we substitute $3(a-1)+b$ for the entry (a,b) in $A \otimes A$. The result is the matrix L. Any pair of entries at centrally symmetric cells in A sum to 4. Therefore, the above substitution implies that any pair of entries at centrally symmetric cells in $A \otimes A$ sum to 10.

Similarly, we may compute $B \otimes B$ and perform the same substitution and the outcome is the matrix R. Again, any pair of entries at centrally symmetric cells in $B \otimes B$ sum to 10.

We also note that $(A \otimes A)P_9 = AP_3 \otimes AP_3 = B \otimes B$. Consequently, The properties of L' and R' described in subsection 2.2 follow.

Acknowledgment The author is grateful to Yaokun Wu for introducing him to the presentation of Hong-Yeop Song [**6**] from which he first learned about Choe's remarkable square.

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6. Hong-Yeop Song, Choi's orthogonal latin squares is at least 67 years earlier than Euler's, A presentation to the 2008 Global KMS Conference, Jeju, Korea.

Summary Orthogonal Latin squares have been known to predate Euler in Europe. However, it is surprising that an Euler square of order nine was already in existence prior to Euler in the Orient. It appeared in a Korean mathematical treatise written by Choe Sŏk-chŏng (1646–1715). Choe's square has several nice properties that have never been fully appreciated before. In this paper, an analysis of Choe's remarkable square is provided and a method of its construction is supplied.

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The Graph Menagerie: Abstract Algebra and the Mad Veterinarian

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Jessica owns three adorable cats: Boo, Kodiak, and Yoshi. Yoshi, unfortunately, has a bad habit: He likes to damage Jessica's carpet. Sometimes Jessica wishes she had a machine that would magically change Yoshi into a tidier pet . . . a goldfish, perhaps. Of course, a goldfish is much smaller than a cat, so perhaps Yoshi could instead be turned into two goldfish. Or maybe two goldfish and a turtle? But goldfish and turtles aren't too cuddly; Jessica might regret the change, so she would want the machine to be able to turn two goldfish and a turtle back into a cat.

In the parlance of recreational mathematics, Jessica sometimes wishes she were a *Mad Veterinarian*. Mad Vet scenarios were originally presented by Harris [**7**], who posed questions as to which collections of animals can be transformed by Mad Vet machines into other collections. Recently, such scenarios have been used as the basis of various problem solving and Math Circle activities; see, for instance, [**13**]. In this article we take a different approach, using Mad Vet scenarios to explore the concepts of groups, semigroups, and directed graphs.

We have two main goals in analyzing Mad Vet scenarios. Corresponding to any Mad Vet scenario there is a naturally defined semigroup, which may or may not be a group. Our first main goal is to help readers gain some intuition about when a given semigroup is actually a group; to this end, we provide a number of not-so-run-of-the-mill examples involving these algebraic structures.

Our second main goal is to illustrate a practice common in mathematics: namely, answering a question in one area by recasting it in another area, answering the recast question there, and then using that result to answer the original question. There are numerous examples of such powerful cross-disciplinary pollination, including Euler's solution to the classic Königsberg Bridges Problem; see, for instance, Chapter 1 in Biggs et al. [**4**]. We provide a beautiful example of this technique, posing an abstract algebraic question and answering it using graph theory.

Along the way, we provide numerous examples and specific computations. We also present some follow-up questions and information which could be used to supplement the material in an abstract algebra course. We assume that the reader is familiar with first-semester abstract algebraic concepts such as groups and equivalence relations. A good source for these topics is Fraleigh [**5**].

1. Mad Vet scenarios

A *Mad Vet scenario* posits a Mad Veterinarian in possession of a finite number of transmogrifying machines, where

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1. Each machine transmogrifies a single animal of a given species into a finite nonempty collection of animals from any number of species;
2. Each machine can also operate in reverse; and
3. There is a one-to-one correspondence between the species with which the Mad Vet works and the transmogrifying machines; moreover, each species' corresponding machine takes as its input exactly one animal of that species.

These three requirements do not explicitly appear in the puzzles posed by Harris [**7**], but they are certainly implicit there.

Let's consider an example.

Scenario #1. Suppose a Mad Veterinarian has three machines with the following properties.

Machine 1 turns one ant into one beaver;

Machine 2 turns one beaver into one ant, one beaver and one cougar;

Machine 3 turns one cougar into one ant and one beaver.

Starting with one ant, the Mad Vet could produce infinitely many different collections of animals. For example, she could use Machine 1 to turn the ant into a beaver, and then use Machine 2 repeatedly to continually increase the number ants and cougars in her collection. Alternatively, she could use Machine 1 followed by Machine 2, and put the resulting cougar into Machine 3, yielding a collection of two ants and two beavers. Then using Machine 1 twice in reverse, she'd obtain a collection consisting of exactly four ants.

We now mathematize these Mad Vet scenarios. Given a scenario involving n distinct species of animals, we let A_i be the species of animal taken as input (in the forward direction) by Machine i, and denote by $d_{i,j}$ the number of animals of species A_j which are produced by Machine i. For example, in Scenario #1, $A_1 =$ Ant, $A_2 =$ Beaver and $A_3 =$ Cougar, and we have, for instance, $d_{1,1} = 0$, $d_{1,2} = 1$, and $d_{1,3} = 0$.

Writing $\mathbb{N}$ for the set $\{0, 1, 2, \ldots\}$ and $\mathbf{0}$ for the trivial vector $(0, 0, \ldots, 0)$ of length n, we define a *menagerie* to be an element of the set

$$S = \mathbb{N}^n \setminus \{\mathbf{0}\}.$$

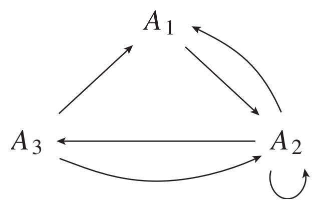
There is a natural bijective correspondence between menageries and nonempty collections of animals from species $A_1, A_2, \ldots, A_n$. For instance, in Scenario #1 a collection of two beavers and five cougars would correspond to $(0, 2, 5)$ in S.

2. Mad Vet graphs

We give here a brief introduction to some standard graph theory concepts. For a more thorough examination of the topic, see, for example, West [**11**] or Wilson and Watkins [**12**]. (Note that graph theory definitions vary widely from text to text; for instance, what we will call a *path* is what West calls a *walk* [**11**].) A *directed graph* consists of a set V of *vertices* and a set E of *edges*; the graph is *finite* if both V and E are finite. Each edge e in E has an *initial vertex*, $i(e)$, and *terminal vertex*, $t(e)$, and is represented in the graph by an arrow pointing from $i(e)$ to $t(e)$. Loops (that is, edges e for which $i(e) = t(e)$) are allowed, as are multiple edges (that is, edges that have a common initial vertex and a common terminal vertex). A vertex is a *sink* if it is not the initial vertex of any edge.

Given any Mad Vet scenario, its corresponding *Mad Vet graph* is the directed graph with $V = \{A_1, A_2, \ldots, A_n\}$, and having, for each A_i, A_j in V, exactly $d_{i,j}$ edges with initial vertex A_i and terminal vertex A_j. Note that any Mad Vet graph is sink-free, due to the third defining feature of a Mad Vet scenario.

EXAMPLE. Scenario #1 has the following Mad Vet graph.



We return to directed graphs in Section 6.

3. Menagerie equivalence classes

Now we come to the key idea. In the context of a Mad Vet scenario, there is a relationship between various menageries. Clearly, a set consisting of two ants and a cougar is different from a set consisting of an ant and three beavers. But if the vet has machines that can be used to replace the first collection of animals with the second (and vice versa), it would make sense to somehow identify the menageries $(2, 0, 1)$ and $(1, 3, 0)$ in S. We have here a naturally arising relation $\sim$ on S, defined formally as follows. Given $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ in S, we say that a is *related to* b, and write $a \sim b$, if there is a sequence of Mad Vet machines that will transmogrify the collection of animals associated with menagerie a into the collection of animals associated with menagerie b. Using the three properties of a Mad Vet scenario, it is straightforward to show that $\sim$ is an equivalence relation on S. The equivalence class of a in S under $\sim$ is

$$[a] = \{b \in S : b \sim a\};$$

such equivalence classes partition S.

We now focus on the set

$$W = \{[a] : a \in S\}$$

of equivalence classes of S under $\sim$. Though the elements of W are actually sets themselves, we will work with them primarily as individual elements of the set W.

EXAMPLE. Suppose that our Mad Vet of Scenario #1 starts with the menagerie $(1, 0, 0)$, that is, a collection consisting of just one ant. Then $(1, 0, 0) \sim (0, 1, 0)$ (using Machine 1); in fact, our previous discussion shows that

$$(1, 0, 0) \sim (0, 1, 0) \sim (1, 1, 1) \sim (2, 2, 0) \sim (4, 0, 0).$$

Using equivalence class notation, we've shown

$$[(1, 0, 0)] = [(0, 1, 0)] = [(1, 1, 1)] = [(2, 2, 0)] = [(4, 0, 0)],$$

that is, that these five expressions all represent same element of W.

Now, let (a, b, c) be any menagerie in this Mad Vet scenario. We claim that (a, b, c) is equivalent to one of the menageries $(1, 0, 0)$, $(2, 0, 0)$, or $(3, 0, 0)$. If $c > 0$, then using Machine 3 c times we see that $(a, b, c) \sim (a + c, b + c, 0)$; then if $b + c > 0$, we can use Machine 1 in reverse $b + c$ times to show that $(a + c, b + c, 0) \sim (a + b + 2c, 0, 0)$. By the transitivity of $\sim$, we conclude that $(a, b, c) \sim (i, 0, 0)$ for some positive integer i (namely, $i = a + b + 2c$). We noted above that $(1, 0, 0) \sim (4, 0, 0)$, which implies that $(2, 0, 0) \sim (5, 0, 0)$, $(3, 0, 0) \sim (6, 0, 0)$, and, more generally, that $(j, 0, 0) \sim (i, 0, 0)$ for any positive integers i and j that are congruent modulo 3. Thus, the only elements of W are

$$[(1, 0, 0)], [(2, 0, 0)], \text{ and } [(3, 0, 0)].$$

We now rule out any redundancy among these three elements of W. Given a menagerie $m = (a, b, c)$, define the sum $s_m = a + b + 2c$. If we apply Machine 1 to m, we obtain menagerie $x = (a - 1, b + 1, c)$; if we apply Machine 2 to m we obtain $y = (a + 1, b, c + 1)$; finally, if we apply Machine 3 to m we obtain $z = (a + c, b + c, 0)$. Since

$$s_x = (a - 1) + (b + 1) + 2c = s_m = (a + c) + (b + c) = s_y$$

and

$$s_z = (a + 1) + b + 2(c + 1) = s_m + 3,$$

we have that if menageries m and n are related under $\sim$ then s_m and s_n are congruent modulo 3. Since $s_{(1,0,0)} = 1$, $s_{(2,0,0)} = 2$ and $s_{(3,0,0)} = 3$, the equivalence classes of menageries $(1, 0, 0)$, $(2, 0, 0)$ and $(3, 0, 0)$ under $\sim$ are all distinct. Hence, for this Mad Vet scenario, W is the 3-element set

$$\{[(1, 0, 0)], [(2, 0, 0)], [(3, 0, 0)]\}.$$

4. Mad Vet semigroups

We can gain some understanding of a Mad Vet scenario by studying its collection, W, of menagerie equivalence classes simply as a set. But we can learn even more if we exploit a natural operation which combines menageries. We first remind the reader of some definitions.

Let S be any set, and let $*$ be a binary operation on S. Recall that $(S, *)$ is a *semigroup* if $*$ is associative; a semigroup $(S, *)$ is a *monoid* if it contains an identity element for $*$; and a monoid is a *group* if each of its elements has an inverse under $*$.

Three important types of semigroups arise in the context of Mad Vet scenarios. First, given a scenario, we have its set S of menageries, equipped with the usual addition of vectors. (Such addition is an acceptable semigroup operation on S since it is associative and since the sum of two nonzero vectors is again nonzero.) Next, we have the scenario's *Mad Vet semigroup*, which we discuss in this section. Finally, we introduce *graph semigroups* in Section 7.

To create the Mad Vet semigroup of a Mad Vet scenario, we define addition on the scenario's set W of equivalence classes of menageries by setting

$$[x] + [y] = [x + y],$$

where addition on the right-hand side of the equation takes place in S. Addition on W can be understood as follows. Suppose a Mad Vet has a collection of animals in her lab corresponding to menagerie x, and is given a new collection of animals corresponding to menagerie y. Then the sum $[x] + [y]$ in W is the equivalence class of the menagerie corresponding to the union of the animals in the two collections.

Since the elements of W are equivalence classes, we must make sure that our addition on W is well defined. But this is straightforward to see, by identifying our menageries with their associated collections of animals: If some sequence of machines transforms menagerie x into menagerie x', and some sequence transforms menagerie y into menagerie y', then these machines, used in tandem, transform menagerie $x + y$ into menagerie $x' + y'$.

Associativity of $+$ on W is inherited from the associativity of $+$ on S. Thus, $(W, +)$ is a semigroup, called the *Mad Vet semigroup* of its corresponding Mad Vet scenario. Since addition is clearly commutative on S, every Mad Vet semigroup $(W, +)$ is commutative.

EXAMPLE. We revisit Scenario #1 and examine its Mad Vet semigroup $(W, +)$. We showed previously that in this case W is the 3-element set

$$W = \{[(1, 0, 0)], [(2, 0, 0)], [(3, 0, 0)]\}.$$

Using the operation $+$ in W, we get, for instance,

$$[(1, 0, 0)] + [(1, 0, 0)] = [(1 + 1, 0, 0)] = [(2, 0, 0)],$$

as we'd expect. But perhaps it's a bit surprising that

$$[(1, 0, 0)] + [(3, 0, 0)] = [(4, 0, 0)] = [(1, 0, 0)].$$

In other words, $[(3, 0, 0)]$ behaves like an identity element with respect to the element $[(1, 0, 0)]$ in W. In fact, $[(i, 0, 0)] + [(3, 0, 0)] = [(i, 0, 0)]$ for any $1 \le i \le 3$. So for this Mad Vet scenario the Mad Vet semigroup $(W, +)$ is a monoid, with identity $[(3, 0, 0)]$. Further, since

$$[(1, 0, 0)] + [(2, 0, 0)] = [(3, 0, 0)]$$

in W, every element in $(W, +)$ has an inverse. Therefore, $(W, +)$ is in fact a group; since its order is 3, it must be isomorphic to the group $\mathbb{Z}_3$.

5. Not all Mad Vet semigroups are groups

Perhaps it is not surprising that the Mad Vet semigroup of Scenario #1 is a group, in light of the explicit description of its elements. In many Mad Vet scenarios, $(W, +)$ is indeed a group; however, we will later see a Mad Vet semigroup that is not even a monoid. Notably, given any Mad Vet semigroup W, the "obvious" choice, $[\mathbf{0}]$, for an identity element of W is not even contained in W, since $\mathbf{0}$ is not in S.

Scenario #2. Suppose the same Mad Vet has replaced two of her machines with new machines.

Machine 1 still turns one ant into one beaver;

Machine 2 now turns one beaver into one ant and one cougar;

Machine 3 now turns one cougar into two cougars.

In this situation W is a monoid, but not a group. First, we claim that

$$W = \{[(i, 0, 0)] \,:\, i \in \mathbb{Z}^+\} \cup \{[(0, 0, 1)]\},$$

where $\mathbb{Z}^+$ denotes the set of positive integers. Indeed, let (a, b, c) be a menagerie for this scenario. If $a = b = 0$ (that is, there are only cougars in the menagerie) then

$c - 1$ applications of Machine 3 yields that $(0, 0, c) \sim (0, 0, 1)$. Else, suppose that at least one of a or b is nonzero. Since $(a, b, c) \sim (a + b, 0, c)$ (using Machine 1 in reverse b times), we may assume that the menagerie contains at least one ant and no beavers. If $c = 0$, then we are done. If $c \neq 0$, then we can apply Machine 3 in the appropriate direction $|a - c|$ times, obtaining a menagerie that contains a ants and a cougars; thus, $(a, 0, c) \sim (a, 0, a)$. Then applying Machine 2 in reverse a times yields $(a, 0, a) \sim (0, a, 0)$, which is equivalent to $(a, 0, 0)$ (using Machine 1).

Hence, W consists of the indicated elements. We may now use arguments similar to the argument utilized in studying Scenario #1 to show that these elements are all distinct in W. This establishes our claim.

The same sorts of computations as before show that $[(0, 0, 1)]$ is an identity element for this Mad Vet semigroup, and hence W in this case is a monoid. But W is not a group, because, for instance, there is no element $[x]$ in W for which $[(1, 0, 0)] + [x] = [(0, 0, 1)]$.

Given a Mad Vet scenario, we can pose a variety of questions regarding the structure of its Mad Vet semigroup. For instance, is its semigroup finite or infinite? Is it a monoid? If it is a monoid, is it a group? Note that if it is a group, then that group is necessarily abelian (since all Mad Vet semigroups are commutative)—but is it necessarily cyclic?

To give some sense of just how diverse Mad Vet semigroups can be, we provide below five additional Mad Vet scenarios (Scenarios #3–7) which include, in some order, a scenario for which (1) W is an infinite group; (2) W is a finite noncyclic group; (3) W is a finite nonmonoid; (4) W is a finite cyclic group, not isomorphic to $\mathbb{Z}_3$; and (5) W is an infinite nonmonoid.

In fact, these five different structures even arise in scenarios where the Mad Vet has just three species in her lab. Our readers are encouraged to try their hands at matching the above-described scenarios with those of Scenarios #3–7. Teachers can also find a sample Mad Vet homework assignment, appropriate for a first-semester abstract algebra course, at the MAGAZINE website. Descriptions of the semigroups arising in the following five Mad Vet scenarios are provided at the end of the article, so that readers can check their work.

Scenario #3.

Machine 1 turns one ant into one beaver and one cougar;

Machine 2 turns one beaver into one ant and one cougar;

Machine 3 turns one cougar into one ant and one beaver.

Scenario #4.

Machine 1 turns one ant into two ants;

Machine 2 turns one beaver into two beavers;

Machine 3 turns one cougar two cougars.

Scenario #5.

Machine 1 turns one ant into one beaver and one cougar;

Machine 2 turns one beaver into one ant and one beaver;

Machine 3 turns one cougar into one ant and one cougar.

Scenario #6.

Machine 1 turns one ant into one beaver;
Machine 2 turns one beaver into one cougar;
Machine 3 turns one cougar into one cougar.

Scenario #7.

Machine 1 turns one ant into one ant, one beaver and one cougar;
Machine 2 turns one beaver into one ant and one cougar;
Machine 3 turns one cougar into one ant and one beaver.

Given the varied properties of Mad Vet semigroups displayed thus far, one may wonder how one can possibly identify when Mad Vet semigroups are groups. In the next section, we translate this algebraic question into a comparable graph-theoretical question, whose solution is used to obtain an answer in the algebraic realm.

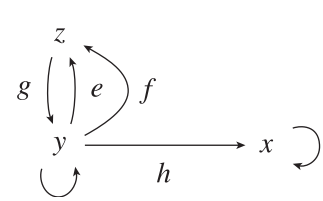
6. The Mad Vet Group Test

In this section, we answer the question: **Given a Mad Vet scenario, when is its Mad Vet semigroup *W* actually a group?**

We need a bit more (standard) graph theory terminology. A *path* in a directed graph Γ is a sequence $P = e_1e_2\cdots e_m$ of one or more edges in Γ for which $t(e_j) = i(e_{j+1})$ for each $1 \le j \le m-1$; we say that P is a path *from* $i(e_1)$ *to* $t(e_m)$. If v and w are vertices in Γ, we say v *connects to* w in case either $v = w$ or there is a path in Γ from v to w. More generally, if $P = e_1e_2\cdots e_m$ is any path in Γ and v is any vertex in Γ, we say v *connects to* P in case v connects to $i(e_j)$ for some edge e_j of P, $1 \le j \le m$. For a vertex v in V, a *cycle based at* v is a path $e_1e_2\cdots e_m$ from v to v for which the vertices $i(e_1), i(e_2), \ldots, i(e_m)$ are distinct. A loop at a vertex is therefore a cycle, with $m = 1$.

The following graph-theoretic definitions might be more unfamiliar to a reader. A finite graph Γ is *cofinal* in case every vertex v of Γ connects to every cycle and to every sink in Γ. Next, if $C = f_1f_2\cdots f_m$ is a cycle in Γ, then an edge e is called an *exit for* C if $i(e) = i(f_j)$ for some $1 \le j \le m$, and $e \ne f_j$. (Intuitively, an exit for C is an edge e, not included in C, which provides a way to momentarily "step away" from C.)

EXAMPLE. Consider the following graph.



The cycle eg based at y has three different exits: f, h and the loop at y. These same three edges are also exits for the cycle ge based at z. Similarly, the loop at y has exits e, f and h. On the other hand, the loop at x has no exit. Also, notice that this graph is not cofinal, since, for example, vertex x does not connect to the cycle eg.

Now we are ready to answer the main question of this section.

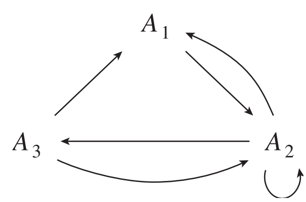
MAD VET GROUP TEST. *The Mad Vet semigroup W of a Mad Vet scenario is a group if and only if the corresponding Mad Vet graph Γ has the following two properties.*

(1) Γ *is cofinal; and*

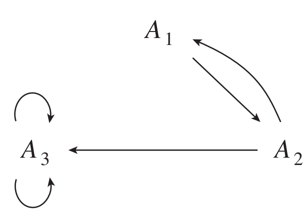
(2) *Every cycle in* Γ *has an exit.*

The proof of this test is too long for this article; however, in Section 7 we will show how the result follows from a more general theorem (whose complete proof is provided in a supplement at the MAGAZINE website). Here, we see how this test applies to some Mad Vet scenarios.

EXAMPLES. Consider again the Mad Vet graph Δ associated with Scenario #1.



By inspection we see that Δ is cofinal (there are no sinks in Δ and every vertex connects to each of the cycles in Δ) and that every cycle in Δ has an exit. Thus the Mad Vet Group Test reconfirms that the Mad Vet Semigroup for this scenario is indeed a group, a fact we established directly in Section 4. On the other hand, recall the Mad Vet graph Θ of Scenario #2.



We see that Θ is not cofinal, since vertex A_3 does not connect to the cycle $A_1 A_2 A_1$. So the Mad Vet Group Test reconfirms that the Mad Vet semigroup of Scenario #2 is not a group, as we saw in Section 5.

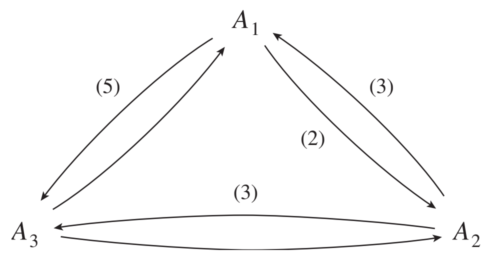
Scenario #8. Consider the Mad Vet scenario described by Harris [**7**], in which the Mad Vet has three machines with the following properties.

Machine 1 turns one cat into two dogs and five mice;

Machine 2 turns one dog into three cats and three mice;

Machine 3 turns one mouse into a cat and a dog.

This scenario has the following Mad Vet graph, where $A_1 =$ Cat, $A_2 =$ Dog, and $A_3 =$ Mouse. The label (d) on an edge e indicates that there are actually d edges in the graph from $i(e)$ to $t(e)$.



It is straightforward to see that this graph satisfies the two properties enumerated in the Mad Vet Group Test; thus, the Mad Vet semigroup in this case is a group, which we identify in Section 8.

You may now want to draw the Mad Vet graphs of Scenarios #3–7, and use the Mad Vet Group Test to determine (or confirm) which three of those Mad Vet scenarios produce Mad Vet groups. Here's one additional observation about the Mad Vet graphs of the remaining two scenarios: One of the graphs is cofinal but contains a cycle without an exit, and the other is not cofinal, though each of its cycles has an exit.

7. Explanation of the Mad Vet Group Test

With the Mad Vet Group Test in hand, we have achieved the second main goal of our article: that is, answering an algebraic question using graph theory. But we have not proven the Mad Vet Group Test. We omit its lengthy proof, but note that the result follows from a theorem about *graph semigroups*. In Section 2, we described a natural connection between Mad Vet scenarios and directed graphs. In fact, a tighter connection can be forged. Any directed graph Γ has an associated commutative *graph monoid*, $(M_\Gamma, +)$. (The interested reader can find the specifics of this construction on p. 163 of Ara et al. in [**2**].) It turns out that if $x, y \in M_\Gamma$ with $x + y = 0$, then $x = y = 0$. Thus, the set $W_\Gamma = M_\Gamma \setminus \{0\}$ is closed under $+$, and so $(W_\Gamma, +)$ is a semigroup, called the *graph semigroup* of Γ.

It follows directly from these constructions that given a Mad Vet scenario with Mad Vet semigroup W and Mad Vet graph Γ, the semigroups W and W_Γ are isomorphic. Thus, information about graph semigroups may be brought to bear in a Mad Vet context. In particular, the main question of the previous section can be answered if we can answer the related question: **Given a directed graph Γ, when is its graph semigroup W_Γ actually a group?**

As it turns out, this question about graph semigroups has recently received significant attention in various mathematical research circles. Some of the related research ideas are described in Section 9. Though in this article we are interested only in sink-free graphs, we do not limit ourselves to such graphs in stating the following result.

GRAPH SEMIGROUP GROUP TEST. *Let Γ be a finite directed graph. Then the graph semigroup W_Γ is a group if and only if Γ has the following three properties.*

(1) *Γ is cofinal;*

(2) *Every cycle in Γ has an exit; and*

(3) *Γ contains no sinks.*

Since Mad Vet graphs are sink-free, this test immediately implies the Mad Vet Group Test. The interested reader can find Enrique Pardo's proof of this result at the MAGAZINE website. While Pardo's proof is too long to include here, we note that the Mad Vet Group Test can be proven using only undergraduate-level graph theory and abstract algebra tools.

8. Classification of Mad Vet groups

Though we have achieved our two main goals, another natural question remains: **When a Mad Vet semigroup is a group, just exactly what group is it?** We turn to another area of mathematics—namely, linear algebra—for an algorithmic way of finding the structure of any Mad Vet group. Note that a Mad Vet semigroup must be a *group* in order for this method to apply.

Let Γ be the Mad Vet graph of a Mad Vet scenario whose Mad Vet semigroup is a group. The graph Γ has an associated *incidence matrix* A_Γ, defined as follows: Suppose Γ has n vertices, $v_1, v_2, \ldots, v_n$. Then A_Γ is the $n \times n$ matrix (d_{ij}), where d_{ij} is the number of edges with initial vertex v_i and terminal vertex v_j (for all $1 \leq i, j \leq n$). For example, if Δ is the graph of Scenario #1, then

$$A_\Delta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

First, we form the matrix $I_n - A_\Gamma$, where I_n is the $n \times n$ identity matrix. For instance, using the above matrix A_Δ, we have

$$I_3 - A_\Delta = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Then we put the (square) matrix $I_n - A_\Gamma$ in *Smith normal form*. The Smith normal form of an $n \times n$ matrix having integer entries is a diagonal $n \times n$ matrix whose diagonal entries are nonnegative integers

$$\alpha_1, \alpha_2, \ldots, \alpha_q, 0, 0, \ldots, 0$$

such that α_i divides α_{i+1} for each $1 \leq i \leq q-1$. The Smith normal form of a matrix A can be obtained by performing on A a combination of these matrix operations: interchanging rows or columns, or adding an integer multiple of a row [column] to another row [column]. The resulting Smith normal form of matrix A is thus of the form PAQ, where P and Q are integer-valued matrices with determinants equal to ± 1. Many computer algebra systems have a built-in Smith normal form function.† For more information about the Smith normal form of a matrix, see, for example, Stein [**10**] or Chapter 23 in Hogben [**8**].

Here's a way of answering the "just exactly what group is it?" question.

MAD VET GROUP IDENTIFICATION THEOREM. *Given a Mad Vet scenario whose Mad Vet semigroup, W, is a group, let Γ be its associated Mad Vet graph. Then*

$$W \cong \mathbb{Z}_{\alpha_1} \oplus \mathbb{Z}_{\alpha_2} \oplus \cdots \oplus \mathbb{Z}_{\alpha_q} \oplus \mathbb{Z}^{n-q},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_q$ are the nonzero diagonal entries of the Smith normal form of the matrix $I_n - A_\Gamma$.

The justification of this theorem is beyond the scope of this article, but the very enthusiastic reader can find a similar justification in Section 3 of Abrams et al. [**1**].

†For instance, to use Maple to compute the Smith normal form of a matrix B, define B in Maple, load the package *LinearAlgebra*, and use the command *SmithForm(B)*. A word of caution: the Smith normal form function in some computer algebra systems will not find the Smith normal form of a matrix of determinant 0, even though such a Smith normal form always exists in this case. A matrix of that type may arise in some Mad Vet scenarios; indeed, it arises in one of our eight numbered Mad Vet scenarios.

EXAMPLE. Letting Δ be the Mad Vet graph of Scenario #1, the Smith normal form of the matrix $I_3 - A_\Delta$ is the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Because we already know that Scenario #1's semigroup is a group, the Mad Vet Group Identification Theorem implies that it is isomorphic to $\mathbb{Z}_1 \oplus \mathbb{Z}_1 \oplus \mathbb{Z}_3 \cong \{0\} \oplus \{0\} \oplus \mathbb{Z}_3 \cong \mathbb{Z}_3$, as expected.

See if you can now use this method to identify the three groups which arise among Scenarios #3–7. Finally, try applying this method to Scenario #8; you should get that the Mad Vet group in that case is isomorphic to $\mathbb{Z}_{34}$.

9. Beyond the Mad Vet

By this point, you may be wondering: **Who really cares about Mad Vet semigroups anyway?** Good question! In case you are not convinced that Mad Vet semigroups are of interest in their own right, we present the following theorem. Although this result is rather technical, our point in stating it is to emphasize the fact that Mad Vet semigroups do indeed play a central role in current, active lines of mathematical research. Not only that, but this theorem actually bridges two apparently different branches of mathematics (algebra and analysis) and the Graph Semigroup Group Test is exactly the link between them.

PURELY INFINITE SIMPLICITY THEOREM. *For a finite directed sink-free graph Γ, the following are equivalent:*

(1) *The Leavitt path algebra $L_{\mathbb{C}}(\Gamma)$ is purely infinite and simple. (This is a statement about an algebraic structure.)*

(2) *The graph C^*-algebra $C^*(\Gamma)$ is purely infinite and simple. (This is a statement about an analytic structure.)*

(3) *Γ satisfies the conditions of the Graph Semigroup Group Test.*

(4) *The graph semigroup W_Γ is a group.*

In the interest of brevity, we have not stated the most general form of this result. Pardo's direct proof of the equivalence of (3) and (4), which involves only undergraduate-level graph- and group-theoretic ideas, is new; the only published proof of this equivalence of which the authors are aware involves showing that both (3) and (4) are equivalent to (1). The *very* energetic reader may wish to consult Arando Pino et al. [**3**].

Finally, as promised earlier, here is a description of the Mad Vet semigroups arising in Scenarios #3–7. In order, these scenarios' semigroups are (up to isomorphism) the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, a 7-element nonmonoid, the group $\mathbb{Z}$, the monoid $\mathbb{Z}^+$, and the group $\mathbb{Z}_4$. For details, see our Analyses of Mad Vet Scenarios #3–7, available at the MAGAZINE website.

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Summary In this paper, we explore Mad Veterinarian scenarios. We show how these recreational puzzles naturally give rise to semigroups (which are sometimes groups), and we point out a beautiful, striking connection between abstract algebra and graph theory. Linear algebra also plays a role in our analysis.

GENE ABRAMS received his Ph.D. in Mathematics from the University of Oregon in 1981 under the direction of Frank Anderson. He is pleased to have coauthored this article with a (much younger, much wiser) mathematical sibling! He has been an algebraist at the University of Colorado at Colorado Springs since 1983. He is proud to have been designated as a University of Colorado systemwide President's Teaching Scholar, as well as the 2002 MAA Rocky Mountain Section Distinguished Teaching Award recipient. When not out riding his bicycle, he surrenders to his passions for baseball and the New York Times Sunday Crossword.

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The Ergodic Theory Carnival

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Ladies and gentlemen, children of all ages. Come one, come all, to see the amazing sights at our ergodic theory carnival! Step right up, friends, and we will show you some of the mysteries seen around the carousel and in a taffy pulling booth. We will see a carnival photographer and find out what kinds of carousel rotations work best for her photographs. We will meet a magician who knows how to find a jewel in a pile of taffy without getting his hands sticky!

You've got to see it to believe it, but these situations can be analyzed by an area of mathematics called ergodic theory. That's right, folks, not only will we look at a collection of basic piecewise linear functions that model activities at the carnival, but we will also use ergodic theory to distinguish between these activities. Come right over and watch how very small differences in local behavior cause big differences in the long term behavior of functions!

What else is ergodic theory good for, you ask? Well, let me tell you. You can use it to explain what happens to a system over time. This marvelous mathematics was first used to study statistical mechanics and investigate the motion or flow of gases over time [**7, 10, 15**]. But wait! There's more! For no extra cost you can use ergodic properties in number theory to calculate how frequently any digit occurs in the real-number base $\beta > 1$ expansion of a number in $[0, 1]$ [**3, 12, 15**]. Believe it or not, you can even use ergodic theory in the field of environmental science to assess the validity of ecosystem models for pine forests [**11**]. An ergodic function has the property that if you look long enough at its iterates on an arbitrary point you can obtain information that represents the entire system. Starting at any other point gives you exactly the same information. There is no sleight of hand here, folks; what you see is what you get.

Gather around and watch what we are going to do! Grab some cotton candy, bring your mathematical intuition, and join us for a great show.

Basic examples

As you enter our carnival, stop first at the carousel with its artistically crafted horses and distinctive music. Find a place to stand by the side of the carousel and watch the activity for a while. Notice the photographer taking pictures of children riding on the carousel. She has set up her tripod at the best vantage point, and she takes a picture every time the carousel stops.

As a mathematician, you notice that each movement of the carousel can be described as a function on a circle, ignoring the up-and-down movement of the horses. Pick the horse nearest to you on the edge of the carousel and call its initial point zero. Let the circumference of the carousel be one unit. As the carousel moves, the distance

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of the horse from you, measured along the circumference of the carousel in the direction of motion, increases from 0 to 1. But wait! When it has traveled one unit, it is back at its initial point. The location of any horse at any instant is described by its distance along the edge of the circle in the counterclockwise direction, or a number in [0, 1] where 0 and 1 represent the same location.

Let's practice by describing the motion of the horses while the carousel is stopped to let children on. If a horse starts at the location x, let $I(x)$ be its location at the end of this motion. It isn't moving! So, $I(x) = x$. That was easy!

The operator starts the carousel again and could stop it after it travels any distance. For now, the horse that starts at zero moves halfway around the circle and stops. It has been a very short ride for everyone, and because the carousel is a solid structure, every horse has moved exactly halfway around the circle. Using your mathematical skills, you think of a function to represent this circular motion. While you might consider a function taking the circle to itself, here we represent the distance traveled along the edge of the carousel as a function from [0, 1] to [0, 1]. If a horse starts at x, let $C(x)$ be its location at the end of this motion. Then $C(x)$ is defined by

$$\begin{aligned} C(x) &= \begin{cases} x + 1/2 & \text{if } 0 \le x < 1/2 \\ x + 1/2 - 1 & \text{if } 1/2 \le x < 1 \end{cases} \\ &= (x + 1/2) \bmod 1. \end{aligned}$$

The graph of $C(x)$ is shown in FIGURE 1. This function is often called a *rational rotation of the circle* because we are rotating by the rational number 1/2. Defining $C(x)$ as a piecewise function might seem complicated, but viewing it this way will simplify the ideas presented later.

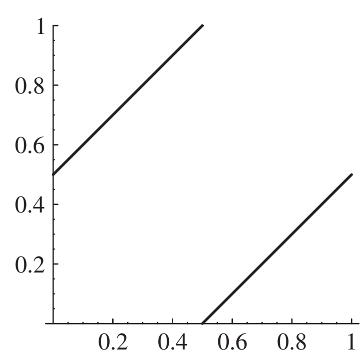

Figure 1 The carousel function $C(x) = (x + 1/2) \bmod 1$

Next to the carousel, you see a booth where two clowns are pulling taffy. You watch while the first clown holds one end of the taffy, while the second clown stretches it to twice its length. Then the second clown folds the taffy over so that the end he was holding is on top of the end that the first clown is holding. FIGURE 2 illustrates this taffy folding method. The second clown then picks up the newly created end at the folded crease, and the process is repeated.

You notice that each step of this process can be described as a function on the interval [0, 1]. Let the first clown be at location zero, and define the original length of the taffy to be one unit. When the second clown stretches the taffy to twice its length and folds it over, his end moves from a place one unit from the first clown to zero units from the first clown. A point in the middle of the original taffy ends up one unit away from the first clown and in the second clown's hands. We can use the map

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/2 \\ -2x + 2 & \text{if } 1/2 \le x < 1 \end{cases}$$

to describe the taffy pull, and the graph of $T(x)$ appears in FIGURE 3. This map is commonly referred to as a *tent map* because of the shape of the graph.

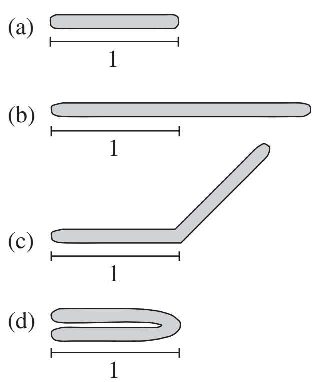

Figure 2 (a) original taffy; (b) stretch the taffy to twice its original length; (c) fold taffy in half; (d) smush taffy

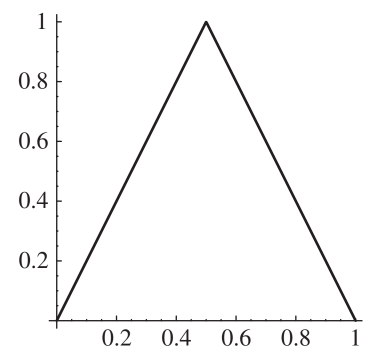

Figure 3 The taffy function $T(x)$

As you watch the repetitive motion of the clowns, a magician appears and, with a sly smile, leans over the taffy and drops a shiny jewel into the sticky mess. It lands about 3/4 of the distance from the first taffy puller toward the second taffy puller and, after one quick stretch and fold, you catch a glimpse of it about halfway between them. The taffy pullers are stretching and folding so quickly that you lose track of the jewel, and you wonder if the magician will be able to find it again.

Invertibility

While you stand there eating your cotton candy and watching the carnival sights, you contemplate how the attractions you have seen are similar and how they are different. You can begin by comparing the properties of the carnival functions we have already defined, $I(x)$, $C(x)$, and $T(x)$.

What would happen if the carousel were rotated in reverse? What if the taffy pullers were to try to undo their work? It is easy to see that $I(x)$ can be reversed. That is, since the horses don't move at all, every horse arriving at $I(x)$ comes from one previous point—in this case, x. Mathematically, this property is called invertibility. A function $f(x)$ is called *invertible* if it is one-to-one, so that for any element y in the range of f, there is exactly one element x in the domain with $f(x) = y$. Even when the carousel rotates, it is certainly possible to undo the rotation, sending each horse backwards to the place where it started. Therefore, the carousel function, $C(x)$, is invertible. (Parents are lucky that the carousel is invertible; if it weren't, reversing the direction of the carousel would take a horse and child back to more than one location—the one he or she originally started from as well as cloned duplicates of the child in other locations. Children are hard enough to keep up with already!)

Attempting to invert $T(x)$, however, is a little more sticky. Notice that $T(1/4) = T(3/4) = 1/2$. That is, applying the taffy function in reverse would take each portion of the taffy, break it into two pieces, and send these pieces to different locations. It becomes a gummy mess, which is what is expected if one attempts to unmix taffy. It also means that the taffy function is not invertible.

Lebesgue measure

Now, we take you on a quick trip away from the midway. Up next, we show you the strange and mystifying sideshow attraction of measure theory. Those with sensitive stomachs should look away as we generalize the concept of length to frightening and grotesque subsets of the real line.

Step right up, ladies and gentleman, young and old, to see the wonderful and mysterious Lebesgue measure. If you have previously seen the secrets of the fantastic integration developed by Henri Lebesgue then you may move immediately to the next section of our carnival. But no one else should miss this attraction!

The familiar Riemann integration that you learned in calculus originated in the work of Newton and Leibniz, and it only works on functions that are relatively nice. In particular, we expect the sets that we use to be no more complicated than countable unions of disjoint intervals contained in $[0, 1]$. Using that the length of an interval $[a, b]$ is $l([a, b]) = b - a$, we can clearly define the length of sets that are countable unions of disjoint intervals. The length of the set is just the sum of the lengths of the intervals. This concept of length is critical to the definition of Riemann integration.

Henri Lebesgue worked to extend the concepts of integration to functions that are much more bizarre. He did this by generalizing the notion of length to what is called a *measure* that is defined on more complicated sets. We now offer entire classes (one may be starting soon in the Chautauqua tent right over there!) on the theory of measurable sets, measures, and integration, and mathematicians are still conducting interesting research in these areas. Here, we sketch an outline of the development of Lebesgue measure, the details of which can be found in Halmos' book on measure theory [**6**].

To begin, we define *Lebesgue outer measure* τ, which is a function defined on all subsets E of $[0, 1]$. First, we take a countable collection of open intervals whose union contains the set E and find the sum of the lengths of the intervals in that collection. Then we take the greatest lower bound of the lengths over all such unions of open intervals containing E. This serves to minimize any overlap and measure E as closely as possible. The greatest lower bound is called the *outer measure* of E, or $\tau(E)$.

Now, we really want to have the relationship $\tau(E) + \tau([0, 1] \setminus E) = 1$. That is, E and its complement should surely combine to have the length of $[0, 1]$, and no more. That's just common sense! When that happens, we say that E is a *Lebesgue measureable set*, and we define the *Lebesgue measure* of E, $\mu(E)$, to be $\mu(E) = \tau(E)$. If $[a, b]$ is an interval, then $\mu([a, b]) = b - a$, as we expect. So Lebesgue measure is a generalized length function that can be applied to more complicated subsets of $[0, 1]$.

But not all subsets! Unfortunately, there are some complicated subsets of the interval (sideshow horrors, unsuitable for most visitors) for which Lebesgue outer measure gives rise to some paradoxes that conflict with properties that we expect any length function to have, so Lebesgue outer measure is not really a length function. Do you want to enter the Sideshow of Strange Pathologies? No, no, turn back! The uninitiated may be shocked by the behavior of sets born from the Axiom of Choice. Skip the next two paragraphs!

We will now construct for you a set that has no Lebesgue measure. The first step is to suppose that x and y are two numbers in $[0, 1]$ and define x to be equivalent to y

if and only if $x - y$ is a rational number. That seemingly tame axiom we mentioned allows us to conjure up a subset of [0, 1] that contains exactly one element from each equivalence class; call it $\mathcal{N}$. For each rational number r in [0, 1], we define another subset $\mathcal{N}_r$ as follows:

$$\mathcal{N}_r = \{x + r : x \in \mathcal{N} \cap [0, 1 - r)\} \cup \{x + r - 1 : x \in \mathcal{N} \cap [1 - r, 1]\}.$$

This slight-of-hand moves $\mathcal{N}$ r units to the right, and then moves the part that extends beyond the point 1 backwards by 1 unit. It takes a little bit of work, but it is not difficult to show that [0, 1] is the disjoint union of the sets $\mathcal{N}_r$.

The length of $\mathcal{N}_r$ for each r should be the same because they are just translations of $\mathcal{N}$, but the sum of the lengths of the $\mathcal{N}_r$s over the countably infinite rational numbers in [0, 1] must be the length of the entire interval. If each $\mathcal{N}_r$ has a positive length then the sum would be infinite, contradicting your knowledge that the length of [0, 1] is one. Similarly, if each $\mathcal{N}_r$ has length 0, then the sum would be 0. This is a paradox, and we end up with a strange and alarming set whose length cannot be measured.

Welcome back, and for the sake of your sanity, be glad that you skipped the last two paragraphs!

Measure-preserving functions

Back in the safety of the midway, we return to comparing the properties of the functions that we have seen. How do the carousel and taffy pull treat our new friend Lebesgue measure? Do children change in size? Does the amount of taffy shrink? Mathematically, we are asking whether the functions preserve measure. To introduce the formal definition, we need to define the preimage of the set A as the set $f^{-1}(A) = \{x : f(x) \in A\}$.

DEFINITION 1. *If μ is Lebesgue measure on* [0, 1] *and $f : [0, 1] \to [0, 1]$ is a function, then f preserves the measure μ if $\mu(f^{-1}(A)) = \mu(A)$ for every measurable set A.*

If we consider the identity map, the inverse image of any measurable set A is simply itself, $I^{-1}(A) = A$, so it easily follows that I preserves measure.

The carousel function, $C(x)$, simply rotates every point x to a location halfway around the carousel, and $C^{-1}(x)$ rotates the carousel halfway the other direction. If we see a certain number of children on the carousel right now, there were the same number there before the carousel rotated. The children did not multiply or disappear. The measure of any set of children is not changed by $C^{-1}(x)$, and therefore $C(x)$ is measure-preserving. Even if we modify $C(x)$ to rotate by an amount other than 1/2, $C_a(x) = (x + a) \bmod 1$ for some real number a, Lebesgue measure is preserved because $C_a(x)$ is still just a translation. FIGURE 4 shows the graph of one example of a modified carousel function, $C_{\sqrt{2}/2}(x) = (x + \sqrt{2}/2) \bmod 1$.

Unlike $C(x)$, the taffy function $T(x)$ is not a simple translation, so it may appear as if this function does not preserve measure. This function is not even invertible! But, does $T(x)$ preserve measure?

When the taffy is pulled, the original slab of taffy is stretched to twice its length. Then the taffy is folded over to make a new piece of taffy the same length as the original piece. If we reverse the process, any set A in the interval [0, 1] of taffy has to be unfolded into two pieces, and then each piece is shrunk to half of its length (yes, this would be difficult to do in real life!). FIGURE 5 illustrates this procedure. Since each piece is reduced by half its length and there are two pieces, the pre-image of A has the

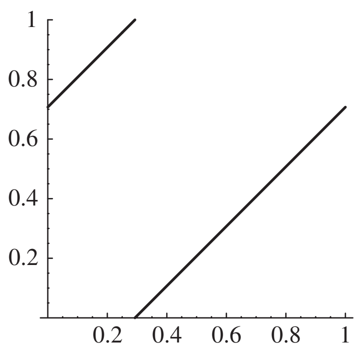

Figure 4 The modified carousel function $C_{\sqrt{2}/2}(x) = (x + \sqrt{2}/2) \bmod 1$

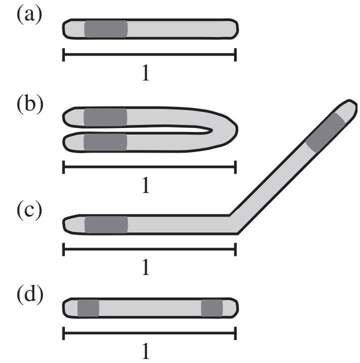

Figure 5 (a) select part of the taffy; (b) un-smush taffy; (c) unfold taffy; (d) shrink taffy back to original length

same measure as A. Looking at this more formally, imagine having an interval $[a, b]$ in $[0, 1]$. Then, $T^{-1}([a, b]) = [\frac{a}{2}, \frac{b}{2}] \cup [1 - \frac{b}{2}, 1 - \frac{a}{2}]$. It follows that the Lebesgue measure of $T^{-1}([a, b])$ is $[\frac{b}{2} - \frac{a}{2}] + [(1 - \frac{a}{2}) - (1 - \frac{b}{2})] = 2(\frac{b}{2} - \frac{a}{2}) = b - a$ which is the measure of $[a, b]$. Since this holds for all intervals and since any measurable set A has a measure based on all intervals containing A, it follows that T is a measure-preserving function. In other words, we don't lose any taffy in the process, and it is spread evenly in each step.

The fact that our taffy function is measure-preserving is based on the fact that when we mix the taffy, any newly mixed piece (interval) comes from two pieces which are each half the length of the new piece. This is directly related to the fact that we stretched the taffy to twice its length. What if we modify the taffy function to allow stretching by a different amount? Suppose that a new taffy-pulling clown arrives at the scene. Instead of stretching the taffy to twice its length, the new clown stretches the taffy from a length of one to a length of $3/2$ and then folds the newly stretched taffy over, making a crease at the point one unit from 0 like before. This time, part of the taffy is not covered by the newly stretched part, and the graph is not symmetric, as seen in FIGURE 6. The new resulting taffy fold function becomes:

$$T_{3/2}(x) = \begin{cases} \frac{3}{2}x & \text{if } 0 \le x < \frac{2}{3} \\ -\frac{3}{2}x + 2 & \text{if } \frac{2}{3} \le x < 1. \end{cases}$$

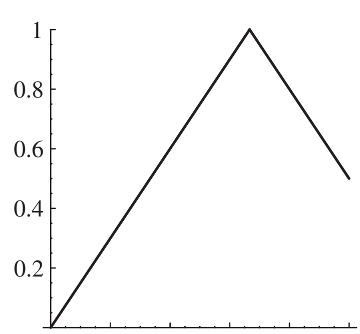

Figure 6 The modified taffy fold function $T_{3/2}$

If we consider a portion of the taffy near 0, say the interval $A = [0, 1/3]$, then the measure of the pre-image of A, $T_{3/2}^{-1}(A)$, is $(1/3)/(3/2) = 2/9$. But the measure of A is $1/3$. Since A is a measurable set, $T_{3/2}$ is not a measure-preserving function, even though no taffy is lost in the process. The main difference is that the taffy is not mixed evenly in this case.

Folks, both our taffy and carousel functions as originally defined preserve Lebesgue measure. However, when we modify the functions, all carousel-like functions preserve Lebesgue measure, but not all taffy-like functions preserve Lebesgue measure.

Ergodicity

Ladies and gentlemen, you are now about to witness the secrets behind ergodicity and how it relates to our carnival functions and modified versions of these functions. We will first show you strange sets that are equal to their preimages.

What does this mean, you ask, to have a set A with $f^{-1}(A) = A$? Watch carefully as our carousel carnies dress two children on opposite sides of our carousel in red clown wigs. Keep your eyes wide open when the operator runs the carousel backwards. That's right, friends, $C^{-1}(x)$ is a rotation halfway around the carousel, so no child ends in the same location as he or she began. But wait! After a half rotation of the carousel backwards, the red wigs are located in exactly the same positions as before, even if the children themselves are in different locations! That's right, folks, if A denotes the set of locations of red clown wigs, we have that $C^{-1}(A) = A$.

Why are sets with $f^{-1}(A) = A$ important? In general, if we have a measurable subset $A \subset [0, 1]$ and measure-preserving function f such that $f^{-1}(A) = A$, then it is also true that $f^{-1}([0, 1] - A) = [0, 1] - A$. In this case, we could simplify things; we could study f by looking at its restriction to A independently from its restriction to $[0, 1] - A$. However, if $\mu(A) = 0$ or $\mu(A) = 1$, then we haven't significantly simplified our study. Functions that cannot be simplified in this way are called *ergodic*. In other words, if the measure of A is strictly between 0 and 1, then for f to be ergodic, it is necessary that f^{-1} moves at least part of the set A to somewhere else.

DEFINITION 2. *If μ is Lebesgue measure on $[0, 1]$ and $f : [0, 1] \to [0, 1]$ is a measure-preserving function, then f is ergodic if the only measurable sets A with $f^{-1}(A) = A$ satisfy $\mu(A) = 0$ or 1.*

Although the carousel function $C(x)$ is measure-preserving, it is not ergodic, and it is very easy to construct a measurable set to verify this. Let $A = [0, 1/4) \cup [1/2, 3/4)$, representing children riding in the first or third quadrants of the circle. Then A is clearly measurable with $\mu(A) = 1/2$, and $C^{-1}(A) = A$, so $C(x)$ is not ergodic. We will see in the next section why this is significant.

What about the other carousel-like functions defined by $C_a(x) = (x + a) \bmod 1$ for a real number a? We know that they are all measure-preserving, but are any of these functions ergodic? If $a = 0$ then we have that $C_0(x)$ is the identity function $I(x)$, and in this case $I^{-1}(A) = A$ for any set A, so $I(x)$ is clearly not ergodic. If the translation number a is any other rational number, then C_a is also not ergodic. For when a is rational, $a = p/q$ for some integers p and q with $q \neq 0$, and p and q have no common factors besides 1. Define $A = [0, \frac{1}{2q}] \cup [\frac{1}{q}, \frac{3}{2q}] \cup [\frac{2}{q}, \frac{5}{2q}] \cup \cdots \cup [\frac{q-1}{q}, \frac{2q-1}{2q}]$. Then $C_a^{-1}(A) = A$, but the measure of A is $1/2$.

If the translation number a of $C_a(x)$ is irrational, then we are in a much different situation. If a is irrational, then $C_a(x)$ is ergodic. While this is difficult to prove rigorously from the definition, it is not too challenging to see why the conditions of ergodicity must hold on intervals. Suppose that the set A contained an interval $[c, d]$. We know that no matter how many times we run the carousel, we always end up with a set of length $d - c$. Since $C_a^{-1}(A) = A$, it follows that $C_a^{-1}([c, d]) \subset A$. Using the same reasoning, $C_a^{-1}(C_a^{-1}([c, d])) \subset A$, and so on. However, since a is an irrational number, the points $C_a^{-n}(c) = C_a^{-1} \circ C_a^{-1} \circ \cdots \circ C_a^{-1}(c)$, where we perform n compositions, fill out the circle. That is, no matter where you decide to stand around the carousel, at some time the left endpoint c will stop arbitrarily close to you. If kids with red clown wigs were sitting in the interval $[c, d]$, then no matter where you stand before the carousel moves, at some time there will be a red wig almost directly in front of you; thus there was a red wig in front of you before the carousel moved. So all points on the carousel must belong to A, and $\mu(A) = 1$. Recall that FIGURE 4 shows the graph of a modified carousel function with $a = \sqrt{2}/2$, which we now know is ergodic since the translation number is irrational.

Don't let this sleight of hand fool you into thinking that this is a complete proof that $C_a(x)$ is ergodic when a is irrational! Remember from our earlier discussion that Lebesgue measurable sets are more complicated than intervals, or even infinite unions or intersections of intervals. Still, examining intervals gives us some idea about why C_a is ergodic when a is an irrational number, and you can find a complete proof in standard ergodic theory books, like ones by Petersen [**10**] or Walters [**15**].

What about the taffy function $T(x)$? It, too, is ergodic. Again, we can examine intervals to obtain a glimmer of understanding as to why this is true. Imagine that you used red food coloring to color a visible section of your taffy that belonged to a set A. After unfolding and shrinking, there would be a red piece closer to the first taffy puller and a red piece closer to the second taffy puller, so those regions had to belong to the original set A as well. Continuing this process, we see that if A contains an interval and $T^{-1}(A) = A$, then $\mu(A) = 1$. A rigorous proof can be found in Nicholis' book on Nonlinear Science [**9**].

You might suspect that any measure-preserving, noninvertible function is ergodic, but that is false. In fact, we can easily modify the taffy function to obtain a measure-preserving function that isn't ergodic. Let's suppose that our original taffy stretching clown, who was skilled at stretching the taffy to twice its original length, returns to the booth. However, after the second clown stretches the taffy to twice its length, instead of folding the taffy, he cuts it in half. Then each clown performs his own taffy fold at the midpoint of his own piece. After this is completed, the second clown sticks the two ends of his piece to the fold of the first clown's piece, so they now have a piece of taffy that has length one again, and they repeat the process. We can represent this function with the equation

$$S(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/4 \\ -2x + 1 & \text{if } 1/4 \le x < 1/2 \\ 2x - 1/2 & \text{if } 1/2 \le x < 3/4 \\ -2x + 5/2 & \text{if } 3/4 \le x < 1. \end{cases} \tag{1}$$

See FIGURE 7 for a graph of $S(x)$. Again, if we look at pre-images of any interval, we end up with exactly two pieces of the same length. In addition, it is not difficult to see that the graph in FIGURE 7 can be decomposed into the function on $[0, 1/2]$ and the function on $[1/2, 1]$. Using arguments similar to those for $T(x)$, the function $S(x)$ is non-invertible and measure-preserving. However, S is not ergodic because $S^{-1}([0, 1/2)) = [0, 1/2)$.

So the original taffy function is ergodic but the carousel function is not. However, as we have shown, we can modify the carousel function to obtain one that is ergodic, and we can modify the taffy function to obtain one that is not!

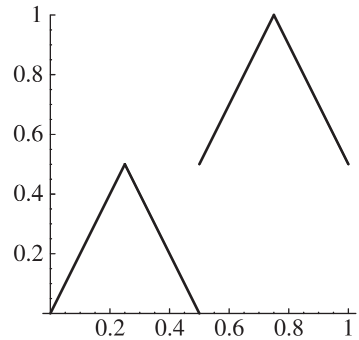

Figure 7 The modified taffy fold function $S(x)$

The ergodic theorem

Folks, you may not yet be convinced that ergodic functions are useful or important, but stick around to see the famous Birkhoff ergodic theorem, proved by George David Birkhoff in 1931 [**2**]. The ergodic theorem ensures that what you observe is representative of the entire system. We will use this theorem to help our photographer, who would like to take pictures of all children on the carousel. If she only takes photos when the carousel stops, which carousel functions will allow her to photograph all of the children? We will also use this theorem to help our magician, who has dropped the jewel into the taffy.

Stick around, friends, and we will show you a simplified version of Birkhoff's ergodic theorem that will resolve the conundrums of our photographer and magician. To do this, we need to define the characteristic function of A,

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

THEOREM 3. (BIRKHOFF'S ERGODIC THEOREM) *If μ is Lebesgue measure on $[0, 1]$ and $f : [0, 1] \to [0, 1]$ is a measure-preserving function, then f is ergodic if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_A(f^n(x)) = \mu(A)$$

for each measurable set A and for almost every $x \in [0, 1]$ (for all $x \in [0, 1]$ except for at most a set of measure 0).

The left-hand side of the equation in Birkhoff's ergodic theorem represents the limit of the average number of times $f(x)$, $f(f(x))$, $f(f(f(x)))$, $\ldots$ lands in the set A. This is commonly known as the *time average*, and the right hand side of the equation is known as the *space average*. In other words, for almost every possible point x, the set $f(x)$, $f(f(x))$, $f(f(f(x)))$, $\ldots$ will eventually land in every set of positive measure, and about as often as the measure of the set would indicate. The statement and proof of Birkhoff's ergodic theorem is beyond the scope of this paper, but we refer the interested reader to Birkhoff's paper, [**2**], or ergodic theory books by Petersen [**10**] or Walters [**15**].

How does the ergodic theorem apply to our photographer, who is taking pictures every time that the carousel stops? If the carousel moves according to the original carousel function $C(x)$, the photographer would photograph the same two children over and over again. This is because $C(x)$ rotates exactly halfway around each time. If we look at $C_a(x)$ for any rational a, she would still see a finite number of children as the day continues. She much prefers the motion x described by $C_a(x)$ when a is irrational. Why? Since we know this system is ergodic, Birkhoff's ergodic theorem implies that almost every point along the edge of the carousel will eventually move into the camera's field of view. The photographer does not have to move, yet she can take photographs of each child if she waits long enough. If she had selected a different location to set up her camera, she would still photograph every child. Hence, when a is irrational, we have a happy photographer.

What about our magician? He simply asks a member of the audience to select one small region on the table to stare at as the taffy pullers work. The magician is convinced that the jewel will reappear in this one location as long as the group waits long enough. Since we showed that the taffy function $T(x)$ is ergodic in the previous section, the ergodic theorem implies that he is correct. However, our magician knows better than to do his jewel trick with the modified taffy function $S(x)$ shown in FIGURE 7. He can't guarantee that the audience member will choose a spot where the jewel will reappear because $S(x)$ is not ergodic.

This brings us back to the question of how ergodic theory is used. In physics, the ergodic theorem implies that studying the motion of a single particle of gas over the long term (the time average) gives the same information as looking at all particles at a particular instant (the space average) [**7, 10, 15**]. Ergodicity is also useful in biomedical signal and image processing. For many tests, such as the electrocardiogram (ECG) and the electroencephalography (EEG), technicians take only one sample recording from a patient and calculate a time average. If the process is ergodic, then they can use the time average to estimate the mean and variance of the signal (the space averages) using the ergodic theorem [**8**]. These examples, my friends, are not just a day at the carnival.

A final question After spending a hot, sticky day at the midway, we want to leave you with one more enticing idea that will compel you to return to our carnival again.

How can we distinguish between the ergodic examples? There are many other properties that play important roles in ergodic theory; we mention one more. We say a function f is *strong mixing* if for all measurable sets A and B

$$\lim_{n\to\infty} \mu(A \cap f^{-n}B) = \mu(A)\mu(B).$$

This means that, in the long run, f distributes B fairly evenly throughout [0, 1]. Strong mixing implies ergodicity, but not all ergodic functions are strong mixing. One of the ergodic examples in this paper is strong mixing with respect to Lebesgue measure, and the other is not. Can you figure out which is which? The answers can be found in the references [**9, 15**].

Acknowledgment The referees made insightful comments and suggestions that greatly improved this paper, and so we novice carnival workers gratefully acknowledge the assistance of our Lot Managers.

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Summary The Birkhoff ergodic theorem, proved by George David Birkhoff in 1931, allows us to investigate the long-term behavior of certain dynamical systems. In this article, we explain what it means for a function to be ergodic, and we present Birkhoff's theorem. We construct models of activities typically found at carnivals and compare and contrast them by analyzing their ergodic theory properties. We use these carnival models to show how Birkhoff's ergodic theorem can be used to help a photographer set up her equipment to take pictures of all children on a carousel and to aid a magician in finding a lost jewel in a sticky mess of taffy.

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Which Surfaces of Revolution Core Like a Sphere?

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A *spherical ring* is the object that remains when a cylindrical drill bit bores through a solid sphere along an axis, removing from the sphere a capsule consisting of a cylinder with a spherical cap on each end, as shown in FIGURE 1. Remarkably, the volume of such a spherical ring depends only on its height, defined as the height of its cylindrical inner boundary, and not on the radius of the sphere from which it was cut.

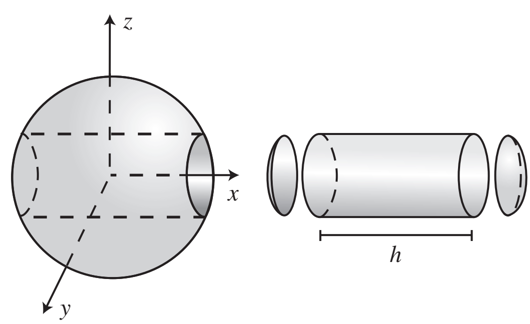

Figure 1 Cutting a spherical ring of height h from a sphere.

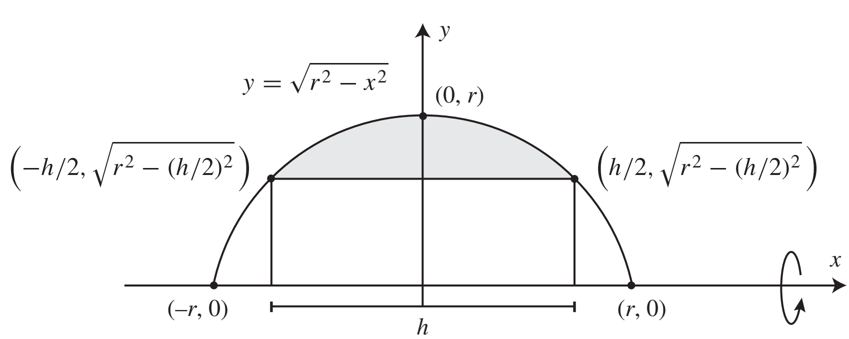

Figure 2 A spherical ring as a solid of revolution.

One straightforward way to verify this fact is to note that all the objects in FIGURE 1 are solids of revolution. This is depicted in FIGURE 2, where everything shown in the xy-plane is to be revolved around the x-axis. There a sphere of radius r is represented

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by the semicircular graph of $y = \sqrt{r^2 - x^2}$, and a spherical ring of height h cut from this sphere is represented by the shaded region below the semicircle and above the horizontal line segment of length h inscribed in the semicircle. We can calculate the volume of this spherical ring by integrating the areas of its annular cross-sections taken perpendicular to the x-axis (the "washer method"):

$$\begin{aligned} V &= \int_{-h/2}^{h/2} \left[\pi(\sqrt{r^2 - x^2})^2 - \pi(\sqrt{r^2 - (h/2)^2})^2\right] \mathrm{d}x \\ &= \pi \int_{-h/2}^{h/2} \left((h/2)^2 - x^2\right) \mathrm{d}x = \frac{\pi h^3}{6}. \end{aligned}$$

At the outset it looks as though V should depend on both r and h, but it turns out to be a function of h only. This is a surprise that challenges many people's intuition. For example, a spherical ring of height one centimeter cut out of a sphere the size of the earth has the same volume as a spherical ring of height one centimeter cut out of a sphere the size of a baseball. How can this be? The reason is that while the inner radius of the ring cut out of the earth is much larger, the radial thickness of this ring is much smaller: about 2×10^{-10} cm, which is less than the diameter of a hydrogen atom. For spherical rings of any fixed height h cut out of spheres of increasing radius r, this tradeoff between increasing inner radius (the quantity $\sqrt{r^2 - (h/2)^2}$ in FIGURE 2) and decreasing radial thickness (the quantity $r - \sqrt{r^2 - (h/2)^2}$ in FIGURE 2) preserves a fixed volume.

This property of the sphere appears in many calculus textbooks as an exercise in calculating volumes of solids of revolution. It has also caught the eye of many recreational mathematicians, perhaps getting its most public airing in the newspaper column of Marilyn vos Savant [**11**]. But, despite its prominence, it seems to lack a name. Since the process of cutting a spherical ring out of a sphere is much like coring an apple, we refer to this property as the *coring property* of the sphere.

Many surfaces of revolution can be similarly cored by cylindrical drill bits centered on their axes of revolution. So it is natural to ask to what extent the coring property characterizes the sphere among surfaces of revolution. Here we pose this question precisely and answer it completely using only elementary ideas from calculus, informed at critical junctures by geometric insight.

The coring property The first order of business is to state the coring property in such a way that it applies to surfaces of revolution other than spheres. The coring property of the sphere compares spheres of different radii r, but each of these is just the unit sphere scaled up or down by the linear scale factor r. So we say that a surface of revolution satisfies the *coring property* if, when the surface is scaled up or down by a linear scale factor and then cored by a cylindrical drill bit centered on its axis of revolution, what remains (exterior to the drill bit) is a ring whose volume depends only on its height, and not on the scale factor. We define a *ring* to be a one-piece solid of revolution having a single cylindrical inner boundary, and the *height* of such a ring to be the height of its cylindrical inner boundary.

To flesh out this formulation of the coring property, and to give us a workable setup for our investigation of it, we need a picture. In general, a surface of revolution S is generated by revolving a plane curve C, called the *profile curve* of S, around a line lying in the same plane as C, which we have already called the *axis of revolution* of S. In particular, a sphere is the surface generated by revolving a semicircle around the line containing its diameter. (In fact, this is how Euclid defined a sphere in his

Elements [**6**, p. 261]!) Since we are essentially generalizing a property of the sphere, we begin with a profile curve looking much like a semicircle, as depicted in FIGURE 3.

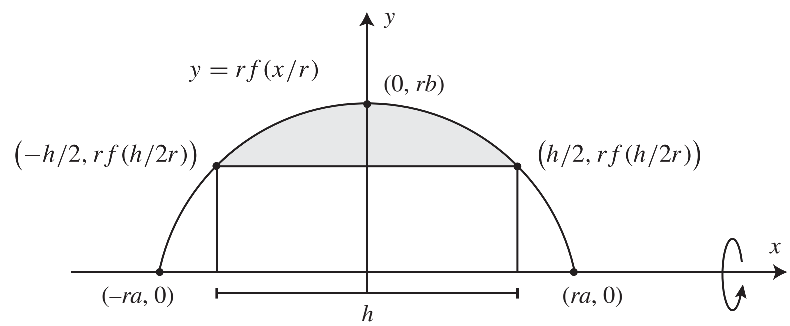

Figure 3 An even profile function $y = f(x)$ scaled by a linear scale factor r.

The profile curve in FIGURE 3 is the graph of an even *profile function* $y = f(x)$ and is to be revolved around the x-axis. We scale the surface S generated by the graph of f by a linear scale factor r, yielding surfaces $S(r)$ generated by the curves $y/r = f(x/r)$, or $y = rf(x/r)$. (For example, if S is a sphere of radius ρ, then $S(r)$ is a sphere of radius $r\rho$.) We can cut a ring out of the solid bounded by $S(r)$ by boring through it with a cylindrical drill bit centered on the x-axis. The resulting ring is generated by revolving the shaded region around the x-axis in FIGURE 3. We say that the surface S satisfies the coring property if the volume $V(r, h)$ of a ring of height h cut out of the solid bounded by $S(r)$ is a function of h alone.

Before striking out in search of surfaces satisfying the coring property, let's examine the assumptions implicit in FIGURE 3, since these will be the hypotheses for any conclusions that we reach based on this picture. To begin with, the profile curve in FIGURE 3 is not self-intersecting and it has exactly two x-intercepts. We accept these assumptions as geometrically natural, because they ensure that the resulting surface S is *closed*: that is, it encloses a single 3-dimensional region.

Two other prominent features of this profile curve are:

1. It is the graph of a function $y = f(x)$.
2. It has a vertical line of symmetry, which conveniently and with no loss of generality is the y-axis.

These assumptions are not quite as cumbersome as they might seem because, for our purposes, the first is subsumed by the second. That is, if a curve C generates a surface that satisfies the coring property and if C is symmetric with respect to the y-axis, then y must be a function of x on C. This is because for any profile curve C that is symmetric with respect to the y-axis on which y is not a function of x, there will be values of h for which two or more rings having the same height h but different volumes can be cut out of the surface generated by C by cylindrical drill bits of different sizes, so that the volume of a ring cannot be a function of its height alone. For example, consider the profile curve C indicated in FIGURE 4. For the value of h indicated there, cylindrical drill bits of radii R_1, R_2, and R_3 will cut rings out of the surface generated by C having the same height h but different volumes. A surface generated by a curve C having a vertical line of symmetry is *centrally symmetric*. That is, it has a *center of symmetry*: a point P (in this case the origin) bisecting every line segment passing through P that connects two points on the surface.

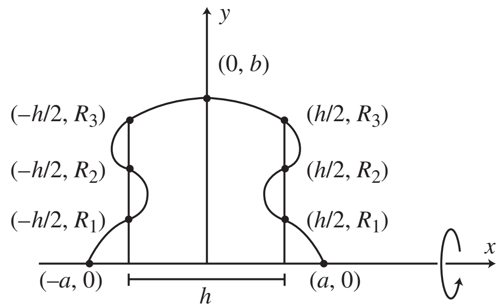

Figure 4 A symmetric profile curve not defined by a function.

So a closed, centrally symmetric surface of revolution S satisfying the coring property must be generated by the graph of an even profile function f having exactly two x-intercepts. In addition, f must be increasing to the left of $x = 0$ and decreasing to the right of $x = 0$, since only then will coring the surface S with a cylindrical drill bit always result in what we have defined to be a ring, which needs to be in one piece. Therefore, to determine which closed, centrally symmetric surfaces of revolution satisfy the coring property, it is safe use FIGURE 3 as a starting point.

The symmetric case: a calculus argument The volume $V(r, h)$ of the ring formed in FIGURE 3 is twice the volume of the right half of the ring, which is the volume enclosed by $S(r)$ on the interval $0 \le x \le h/2$ less the volume of the cylinder drilled out on that same interval:

$$V(r,h) = 2\left(\int_0^{h/2} \pi \left[rf\left(\frac{x}{r}\right)\right]^2 \mathrm{d}x - \pi\left[rf\left(\frac{h}{2r}\right)\right]^2 \frac{h}{2}\right). \tag{1}$$

We wish to identify the functions f for which V depends only on h and not on r. Towards this end, the simplest strategy turns out to be the best: we simply set equal to each other the volumes of two different rings of the same height, and see what we can say about f based on the resulting equation.

In particular, note that for a ring cut out of the unscaled surface S, whose height h will satisfy $0 \le h/2 \le a$, another ring of the same height can be cut out of any scaled-up surface $S(r)$ where $r > 1$, and the volumes of these two rings should be the same. That is, for any h such that $0 \le h/2 \le a$ and any $r \ge 1$, we should have $V(1, h) = V(r, h)$, or from (1):

$$2\left[\int_0^{h/2} \pi\,[f\,(x)]^2\,\mathrm{d}x - \pi\left[f\left(\frac{h}{2}\right)\right]^2 \frac{h}{2}\right]$$
$$= 2\left[\int_0^{h/2} \pi\left[rf\left(\frac{x}{r}\right)\right]^2 \mathrm{d}x - \pi\left[rf\left(\frac{h}{2r}\right)\right]^2 \frac{h}{2}\right] \tag{2}$$

which is easily rearranged to yield

$$\int_0^{h/2} \left([f(x)]^2 - [rf(x/r)]^2\right)\,\mathrm{d}x = (h/2)\left([f(h/2)]^2 - r^2[f(h/2r)]^2\right). \tag{3}$$

For fixed $r \geq 1$, let

$$g(x) = [f(x)]^2 - r^2[f(x/r)]^2.$$

Then for $0 \leq h/2 \leq a$, g satisfies

$$\int_0^{h/2} g(x)\,\mathrm{d}x = \frac{h}{2} g(h/2). \tag{4}$$

Dividing both sides of (4) by $h/2$, we see that the average value of g on any subinterval $[0, h/2]$ of $[0, a]$ is its value at the right endpoint of the subinterval: $g(h/2)$. Does this mean that g must be constant? If f is continuous on the interval $[0, a]$, then so is g, so that both sides of (4) are differentiable functions of h. Differentiating yields

$$\frac{1}{2} g(h/2) = \frac{1}{2} g(h/2) + \frac{h}{4} g'(h/2)$$

so that $g'(h/2) = 0$ for $0 \leq h/2 \leq a$. So indeed, g is constant on $[0, a]$. What is the constant? If, as in FIGURE 3, $f(0) = b$, then

$$g(0) = [f(0)]^2 - r^2[f(0)]^2 = b^2 - r^2 b^2 = (1 - r^2) b^2$$

so that

$$[f(x)]^2 - r^2[f(x/r)]^2 = (1 - r^2) b^2. \tag{5}$$

If, as in FIGURE 3, $f(a) = 0$, then setting $x = a$ in (5) yields

$$[f(a/r)]^2 = \left(1 - \frac{1}{r^2}\right) b^2. \tag{6}$$

This is essentially a formula for f. We can put it in a more recognizable form by making the change of variable $u = a/r$. Since $1 \leq r < \infty$, we have $0 < u \leq a$ and

$$[f(u)]^2 = \left(1 - \frac{u^2}{a^2}\right) b^2.$$

That is, on the graph of f:

$$\left(\frac{y}{b}\right)^2 + \left(\frac{x}{a}\right)^2 = 1. \tag{7}$$

So the graph of f must be a semi-ellipse, which when revolved around the x-axis produces a *spheroid*: a sphere expanded or contracted in the x-direction. Indeed, direct calculation shows that the volume of a ring of height h formed by coring the spheroid of equation (7) is

$$V_{\text{ring}} = \frac{1}{6}\pi \left(\frac{b}{a}\right)^2 h^3$$

which depends only on the shape of the spheroid and on h, and not on the scale of the spheroid. So we have shown:

PROPOSITION 1. *A closed, centrally symmetric surface of revolution generated by a continuous profile curve satisfies the coring property if and only if it is a spheroid.*

The non-symmetric case: a geometric insight To expand our search for closed surfaces of revolution satisfying the coring property, we need to look at surfaces that are not centrally symmetric. But the profile curve of such a surface need not be the graph of a profile function. So how do we describe the profile curves among which we want to search? We must replace FIGURE 3 by the more complicated FIGURE 5.

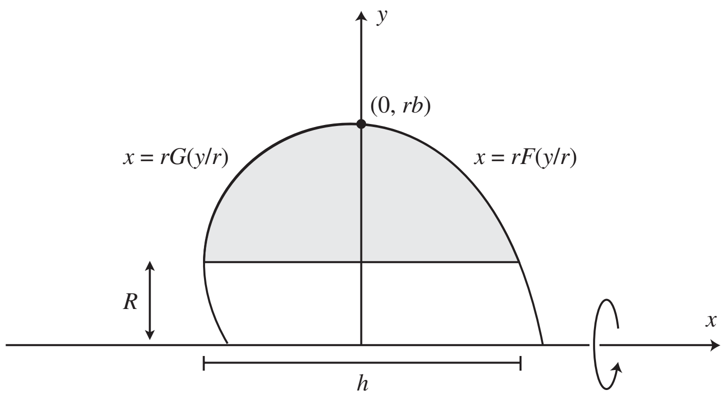

Figure 5 A family of non-symmetric profile curves $C(r)$.

There a non-symmetric profile curve C generating a non-symmetric surface S is scaled by a linear scale factor r to produce a family of profile curves $C(r)$ that generate surfaces $S(r)$. For convenience, we locate the maximum y-value b on the curve C at the point $(0, b)$. Since by hypothesis the curve C has exactly two x-intercepts, one portion of C must connect the rightmost of these x-intercepts with $(0, b)$ and another portion of C must connect the leftmost of these x-intercepts with $(0, b)$. On each of these portions y need not be a function of x, but x is a function of y. Otherwise, coring the surface S with a cylindrical drill bit centered on its axis would not always produce a ring, which by definition has to be in one piece. So the curve C is the union of the graphs of two functions: $x = F(y)$ on the right and $x = G(y)$ on the left. The domain of both F and G is $0 \leq y \leq b$ and $F(b) = G(b) = 0$.

Fortunately, we can reduce this more complicated situation to the simpler one we have already analyzed. We merely *symmetrize* the profile curve C in FIGURE 5 with respect to the y-axis. That is, for each y we horizontally shift the line segment determined by the points $(G(y), y)$ and $(F(y), y)$ on C so that its center is on the y-axis. The left and right endpoints of the shifted line segment then lie the same distance $(F(y) - G(y))/2$ to the left and the right of the y-axis, respectively. This transforms C to the symmetric curve C^* in FIGURE 6. The surface S^* generated by C^* is the *symmetrization* of the surface S generated by C relative to the plane $x = 0$. Clearly S^* is centrally symmetric.

Now suppose we scale both the original curve C and the symmetrized curve C^* by the same linear scale factor r. Coring the resulting surfaces of revolution using the same cylindrical drill bit of radius R centered on the x-axis yields two rings having the same height $h(r) = r(F(R/r) - G(R/r))$. These rings are generated by revolving the shaded regions around the x-axes in FIGURE 5 and FIGURE 6. If the volumes of these rings are calculated using the "shell method", the answer is the same in each case:

$$V = \int_R^{rb} 2\pi y\,(rF(y/r) - rG(y/r))\,\mathrm{d}y.$$

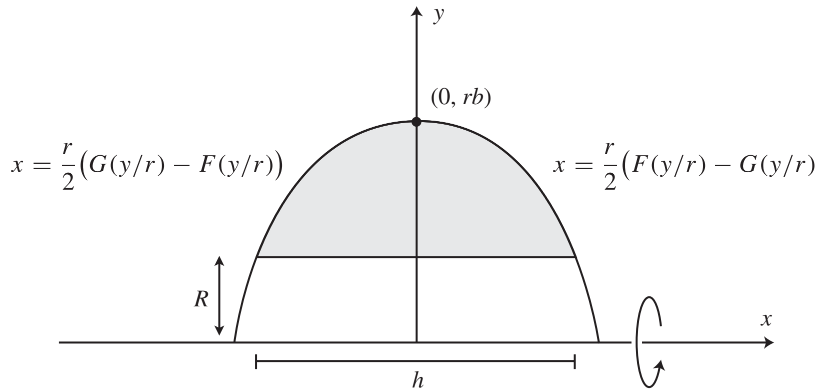

Figure 6 Symmetrized versions $C^*(r)$ of the profile curves $C(r)$.

It follows that the surface S generated by the non-symmetric curve C satisfies the coring property if and only if the centrally symmetric surface S^* generated by the symmetrized curve C^* does. If the curve C is continuous (that is, if F and G are each continuous) then by Proposition 1, S^* satisfies the coring property if and only if it is a spheroid. So we have shown:

PROPOSITION 2. *A closed surface of revolution generated by a continuous profile curve satisfies the coring property if and only if its symmetrization relative to a plane perpendicular to its axis of revolution is a spheroid.*

Examples A variety of surfaces meet the hypotheses of Proposition 2 and therefore satisfy the coring property. The profile curve of each is the upper half of the graph of $(x/a)^2 + (y/b)^2 = 1$ "desymmetrized" by displacing each pair of points sharing a common y value with a horizontal shift that varies continuously with y. For given positive a and b, such profile curves can be produced using either of the following recipes:

1. Choose a continuous "horizontal shift function" $h : [0, b] \to \mathbb{R}$, where $h(b) = 0$ to keep the maximum y-value on the curve at $(0, b)$. Then the profile curve is given by the upper half of the graph of

$$\left(\frac{(x - h(y))}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

2. Choose a right hand portion for the curve: a continuous function $x = F(y)$ where $F : [0, b] \to \mathbb{R}$ and $F(b) = 0$, as in FIGURE 5. Then the left-hand portion of the curve is given by $x = G(y) = F(y) - 2a\sqrt{1 - y^2/b^2}$.

Two profile curves created using the first recipe are shown in FIGURE 7 and FIGURE 8, and two created using the second recipe in FIGURE 9 and FIGURE 10. We have graphed the reflections of these profile curves through the x-axis as well, yielding "side views" of the resulting surfaces of revolution (which we have dubbed the egg, the Star Trek emblem, the acorn, and the heart respectively). In each case $a = b = 1$, so these curves all symmetrize to yield the unit sphere. We have not seen such non-symmetric examples exhibited elsewhere.

Conclusions, Reflections, and Questions Does the coring property characterize the sphere among closed surfaces of revolution? Based on Proposition 2, a fair answer

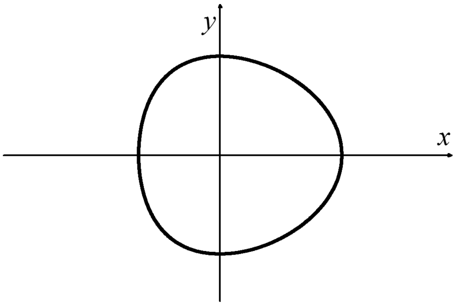

Figure 7 $(x-(1/5)(1-y^2))^2+y^2=1$

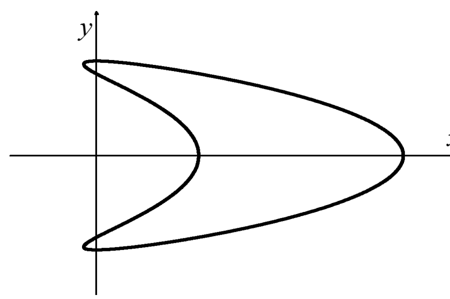

Figure 8 $(x-2(1-y^2))^2+y^2=1$

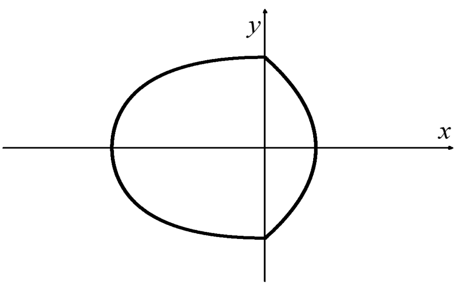

Figure 9 $x=(1/2)(1-y^2)$,
$x=(1/2)(1-y^2)-2\sqrt{1-y^2}$
$(-1\le y\le 1)$

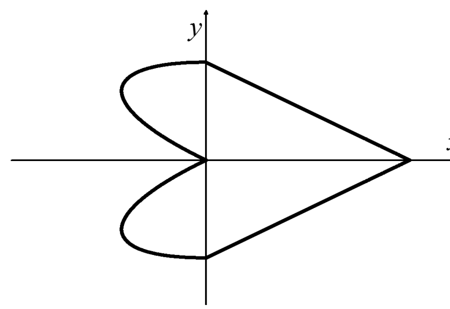

Figure 10 $x=2(1-|y|)$,
$x=2(1-|y|)-2\sqrt{1-y^2}$
$(-1\le y\le 1)$

is: "sort of". Perhaps the largest class of surfaces that are at least vaguely sphere-like are *smooth ovaloids*: surfaces that are *convex*, meaning that the line segment connecting any two points inside the surface is also inside the surface, and *smooth*, meaning that near each point, the surface is the graph of a function having continuous partial derivatives of all orders, so that the surface has no sharp points or edges. Note that the surface in FIGURE 7 is a smooth ovaloid, but the surfaces generated by the profile curves in FIGURES 8–10 are, respectively, smooth but not convex, convex but not smooth, and neither smooth nor convex. The apparent diversity of these surfaces belies their unifying feature: they all yield spheroids when symmetrized.

Our investigation hardly exhausts the topic at hand. There are a number of lesser-known variations on the coring property of the sphere to be found in the literature. In his classic exploration of reasoning by induction and analogy *Mathematics and Plausible Reasoning* [**8**, pp. 190–192 and 201–202], George Polya noted that coring spheres with conical or parabolic drill bits also produces rings whose volumes are determined by their heights alone. Alexanderson and Klosinski have expanded on Polya's observations by presenting an even larger catalog of similar phenomena [**1**].

This discussion may well bring to mind another interesting property of the sphere that can be found in the exercises of almost any calculus text: the fact that the surface area of a zone sliced out of a sphere by two parallel planes depends only on the distance between the planes and not on the location of the zone. Does this "slicing property" characterize the sphere among closed surfaces of revolution? This question was addressed by B. Richmond and T. Richmond in the *Monthly* [**9**], where they named this property the *equal area zones property*. (The sphere turns out to be the only smooth surface of revolution satisfying this property, but some non-smooth surfaces of revolution satisfying this property can also be constructed.) More recently, a generalization of this property involving pairs of surfaces of revolution has been formulated and explored by Cass and Wildenberg [**3**]. Walter Rudin has formulated and examined a variation of the equal area zones property in the context of n-dimensional spheres [**10**]. And more recently, we have examined higher dimensional analogs of both the equal area zones property [**5**] and the coring property [**4**] in the context of more general hypersurfaces of revolution.

Finally, here is a historical question. The machinery of calculus is not required to discover the coring property of the sphere. It can be derived elegantly using Cavalieri's principle [**7**, pp. 206–210]. It can even be cobbled together from the volumes of a sphere, a cylinder, and a spherical cap, all of which were known to Archimedes [**2**, pp. 180–193]. Similarly, the equal area zones property of the sphere follows easily from a proposition of Archimedes (see [**9**]). But we know of no evidence that Archimedes noticed either of these properties. Moreover, it seems to us that it might have been difficult for him to have formulated them given the limitations of the language and notation of his day. Who was the the first to articulate these properties, and when?

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Summary If a cylindrical drill bit bores through a solid sphere along an axis, removing a capsule from the sphere, the object that remains is called a spherical ring. A surprising property of the sphere that is often presented in calculus courses is that any two spherical rings whose cylindrical inner boundaries have the same height also have the same volume, regardless of the radii of the spheres from which they were cut. In this article, we pose and answer the question: to what extent does this property characterize the sphere among surfaces of revolution?

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Coloring and Counting on the Tower of Hanoi Graphs

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The Tower of Hanoi graphs are intricate, highly symmetric, little-known combinatorial graphs that arise from the multipeg generalization of the well-known Tower of Hanoi puzzle. In this paper, we tour this family of graphs, exploring what we and others have shown, and what is open for further investigation. Even a quick glance at FIGURES 1–4 showing the first few examples (which we define more carefully within the paper) suggests patterns waiting to be discovered. We count the order, size, and degrees of vertices and show how alternate methods of counting these objects can be used to derive combinatorial identities. We describe the standard labeling of these graphs, from which we demonstrate that, although these graphs become more complex as their order increases, one measure of their complexity—the chromatic number—remains remarkably simple.

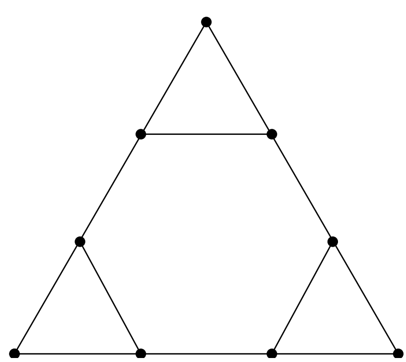

Figure 1 The Hanoi graph H_3^2

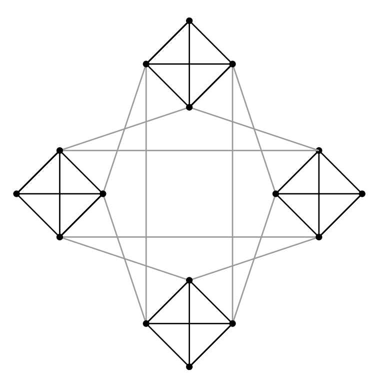

Figure 2 The Hanoi graph H_4^2

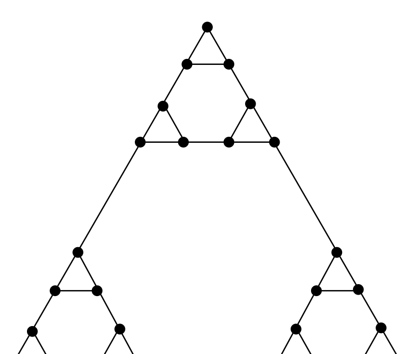

Figure 3 The Hanoi graph H_3^3

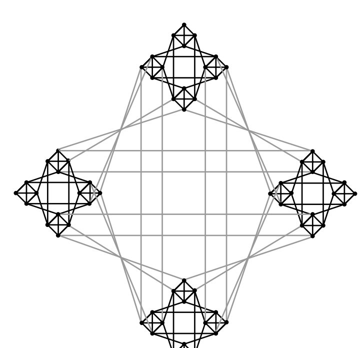

Figure 4 The Hanoi graph H_4^3

Math. Mag. **83** (2010) 200–209. doi:10.4169/002557010X494841.

The Hanoi graphs

The graphs begin with the Tower of Hanoi puzzle. The classic version has three pegs and several disks with distinct diameters, as in FIGURE 5. At the beginning, all of the disks are stacked on the first peg in order by size, with the largest at the bottom. The object is to move the disks so that they are similarly stacked on the second peg. Only one disk may be moved at a time, from the top of one stack to the top of another stack (or onto an empty peg)—and, no disk may ever sit atop a smaller disk. Readers who have never tried the puzzle might wish to play one of the many available online versions.

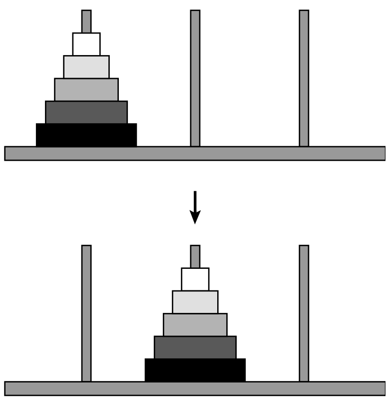

Figure 5 The tower of Hanoi puzzle

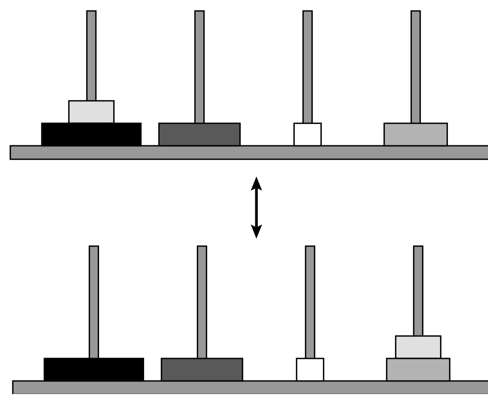

Figure 6 Adjacent states in H_4^5

The puzzle was invented in 1883 by French number theorist and recreational mathematician Édouard Lucas (1842–1891). It was quickly generalized. Lucas himself explored multipeg puzzles as early as 1889. A 4-peg puzzle known as "The Reve's Puzzle" appeared in 1908 in *The Canterbury Puzzles and Other Curious Problems* [**3**]. The problem of counting the number of steps needed to solve the multipeg puzzle (as a function of the numbers of pegs and disks) was posed in 1939 in the *Monthly* [**17**]. Lucas counted the minimum number of moves needed to solve the 3-peg puzzle, but the minimum number of moves needed to solve the 4-peg puzzle has yet to be settled. Of course, if the number of pegs exceeds the number of disks, then the puzzle is trivial, but with each added peg the corresponding graphs become more complicated. Andreas Hinz gives a more detailed history of the puzzle [**4**].

Associated with many puzzles and games is a model called a *state graph*, or *configuration graph*. Its vertices are the legal states, in our case the allowable configurations of disks on pegs. Two vertices are connected by an edge if a single move takes us from one state to the other. The state graph of a Tower of Hanoi puzzle with d disks on p pegs for $p \geq 3$ is called a *generalized Tower of Hanoi graph*, or just *Hanoi graph*, and is denoted H_p^d. These graphs are undirected since every move is reversible.

For example, FIGURE 6 shows two states in the puzzle with five disks on four pegs. We get from the first state to the second by moving the next-to-smallest (light gray) disk from the first to fourth peg. Thus the vertices corresponding to these two states are connected by an edge in the graph H_4^5.

To see how these graphs are built, note that for the (admittedly silly) one-disk puzzle on p pegs, the state graph consists of p vertices with an edge connecting each pair of vertices. That is, $H_p^1 \cong K_p$, the complete graph on p vertices. Another observation for those just getting to know these graphs is that the corners of the large triangle in FIGURE 3 correspond to states with all three disks stacked on a single peg.

For two disks, the subgraph of H_p^2 whose edges correspond to moves of the smaller disk is p disjoint copies of $H_p^1 \cong K_p$. (Each copy of H_p^1 corresponds to a particular fixed placement of the larger disk.) To build the full graph H_p^2, we connect vertices from different components when there is a move of the larger disk between their corresponding states. For example, FIGURE 1 shows the graph H_3^2 built from three copies of the triangle $H_3^1 \cong K_3$, and FIGURE 2 shows the graph H_4^2 built from four copies of the kite $H_4^1 \cong K_4$. Using our imagination, we see H_5^2 built from five copies of the pentagram $H_5^1 \cong K_5$ and so on. We can more easily track this construction using the vertex labeling we present later.

In general, the d-disk graph H_p^d is built from p copies of H_p^{d-1}, each corresponding to a fixed placement of the largest disk, where we connect remote vertices if there is a corresponding move of this largest disk. For example, FIGURE 3 shows the graph H_3^3 built from three copies of H_3^2 and FIGURE 4 shows the graph H_4^3 built from four copies of H_4^2.

This recursive construction suggests that the graphs are *connected*: that we can get from any arrangement of disks on pegs to any other in the puzzle. Though connectedness is not obvious from the puzzle itself, Hinz and Daniele Parisse prove that the Hanoi graphs are not only connected when $p \geq 3$, but also *Hamiltonian*: there exists a cycle visiting each vertex exactly once [**7**]. They also assert that H_p^d is $(p-1)$-*connected*: that the removal of any $p-2$ vertices and their corresponding edges does not disconnect the graph.

The Hanoi graphs for the classic 3-peg puzzle were introduced in 1944 in *The Mathematical Gazette* [**16**]. They bear striking resemblance to Sierpiński's triangles and are a special case of the Sierpiński graphs discussed by various authors [**8, 9, 12, 18**]. They are related to Pascal's triangle, as discussed by David Poole [**15**] and Hinz [**5**]. As an application, Paul Cull and Ingrid Nelson discuss the 3-peg graphs' role in perfect 1-error correcting codes [**2**]. The Hanoi graphs for the puzzle on more than three pegs have been studied since the 1980s, for example by Xiaowu Lu [**13**] and Hinz [**4**].

Though we are interested in the graphs, it is worth mentioning the connection to solving the puzzle. A *path* in a graph is a sequence of distinct vertices, each consecutive pair connected by an edge. The *length* of the path is the number of edges. Solving the puzzle amounts to finding a path from the starting vertex to the ending vertex, and of particular interest are paths of minimal length. In the 3-peg graphs, a minimal path follows the side of the triangle. Hinz and others have expressed hope that understanding the Hanoi graphs might lead to insight on minimal solutions of the puzzle for $p > 3$ pegs.

Counting on the Hanoi graphs

A graph can be measured in many ways, often beginning with the number of vertices, number of edges, and degrees of vertices. In this section, we calculate these quantities for the Hanoi graphs. Then, we derive some combinatorial identities. These results appear (or are implicit) in the work of Sandi Klavžar, Uroš Milutinović, and Ciril Petr [**10**].

How many vertices does H_p^d have? Each of the d disks can be assigned to any of the p pegs. Since disks must be piled largest to smallest on each peg, each assignment produces a unique configuration. Therefore, there are p^d different configurations and, thus, p^d vertices in the graph.

How many edges does H_p^d have? For a fixed pair of pegs, we can move a disk from precisely one of those pegs to the other at every state except where both pegs are empty. Since there are $(p-2)^d$ states with both pegs empty, there are $p^d - (p-2)^d$

states where we can move a disk between this pair of pegs. Each move is counted at each state, which is to say, counted twice. Accounting for our choice of pegs as well, we find the total number of edges is

$$\frac{1}{2}\binom{p}{2}[p^d - (p-2)^d].$$

For example, the graph H_3^3 shown in FIGURE 3 has 27 vertices and 39 edges, and the graph H_4^2 shown in FIGURE 2 has 16 vertices and 36 edges.

Alternatively, for each $1 \le i \le d$, we can move disk i between peg A and peg B as long as none of the $i-1$ smaller disks sit on either of these pegs. There are $\binom{p}{2}$ choices for pegs A and B, p^{d-i} possible placements of the larger disks, and $(p-2)^{i-1}$ placements of the smaller disks. Thus there are

$$\binom{p}{2}p^{d-i}(p-2)^{i-1}$$

edges that correspond to moving disk i. Summing to get the total number of edges and equating with our previous count gives the identity

$$\sum_{i=1}^{d}\binom{p}{2}p^{d-i}(p-2)^{i-1} = \frac{1}{2}\binom{p}{2}[p^d - (p-2)^d].$$

We could have derived this by algebraic manipulation (using the factorization of $x^n - y^n$, where here $x - y = 2$), but is more amusing when it appears from counting on Hanoi graphs.

What is the degree of each vertex? At each vertex there is one incident edge for every pair of pegs, except when both pegs are empty in the corresponding state. Thus, the degree of a vertex corresponding to a state with k occupied pegs, or equivalently k top disks, is

$$\binom{p}{2} - \binom{p-k}{2},$$

where the second term is understood to equal zero if $k = p-1$ or $k = p$.

Alternatively, the only disks that move are top disks, which can move to any other peg unless that peg is occupied by a smaller top disk. Thus, counting from smallest top disk to largest, we find the degree of a vertex corresponding to a state with k occupied pegs equals

$$\begin{aligned}(p-1) + (p-2) + \cdots + (p-k) &= kp - \binom{k+1}{2}\\ &= \binom{p}{2} - \binom{p-k}{2}.\end{aligned}$$

Notice that the degree depends on the number of occupied pegs in the corresponding state. How many states have exactly k occupied pegs? For this count we use the *Stirling number of the second kind*, $S(d,k)$, which equals the number of ways to partition d distinguishable objects into k nonempty subsets. A standard recursion to calculate $S(d,k)$ for $0 \le k \le d$ is

$$S(0,0) = 1; \quad S(d,0) = 0 \quad \text{for } d \ge 1;$$

and

$$S(d, k) = S(d-1, k-1) + kS(d-1, k), \qquad \text{for } d \geq 1.$$

(To see why, note that the first summand counts the partitions where the dth element is in a singleton set.)

Thus we can sort d disks into exactly k nonempty subsets in $S(d, k)$ ways. We can assign these subsets to p pegs in $p(p-1)\cdots(p-(k-1))$ ways; we denote this *falling factorial* by $(p)_k$. Since the subsequent placement of each disk onto its subset's assigned peg is uniquely determined by size, the number of states with exactly k occupied pegs is $S(d, k)(p)_k$.

Klavžar et al. use the Hanoi graphs to derive various combinatorial identities [**10**]. For example, summing over the possible number of occupied pegs and equating our two counts for the total number of vertices give the well-known Stirling identity

$$\sum_{k=1}^{p} S(d, k)(p)_k = p^d$$

for any positive integers d and p.

Similarly, we can compare the number of edges. We count $S(d, k)(p)_k$ vertices corresponding to states with exactly k occupied pegs, each with degree $\binom{p}{2} - \binom{p-k}{2}$. Thus the number of edges in the graph is

$$\frac{1}{2}\sum_{k=1}^{p} S(d, k)(p)_k \left[\binom{p}{2} - \binom{p-k}{2}\right].$$

Equating with our previous count and simplifying give

$$\sum_{k=1}^{p-2} S(d, k)(p)_{k+2} = p(p-1)(p-2)^d,$$

which might appear to be novel but, alas, after canceling $p(p-1)$ reduces to the same Stirling identity for $p-2$.

There are further enumerative uses of the Hanoi graphs. Klavžar et al. showed connections to second order Euler numbers, Lah numbers, and Catalan numbers; they suggest that there may be additional identities available [**11**]. Hinz et al. connect the graphs to Stern's diatomic sequence [**6**].

Labeling and coloring the Hanoi graphs

It is helpful to label each vertex of the Hanoi graph in a way that lets us read off the state of the puzzle it represents. In this section, we describe the standard labeling, which leads to a natural definition of the recursive structure introduced informally earlier and is key to coloring the vertices.

It is customary to number the pegs $0, 1, 2, \ldots, p-1$ and the disks $1, 2, 3, \ldots, d$ from smallest to largest. We say the ith disk sits on peg s_i, for $i = 1, 2, \ldots, d$, and label the vertex corresponding to this state with the string $s_d \cdots s_2 s_1$ in this (reverse) order. Note that the labeling denotes where each disk goes; imagine placing the disks on the pegs, starting with the largest disk and working down by size.

For example, the state shown in FIGURE 7 corresponds to the vertex labeled **173033** in H_8^6.

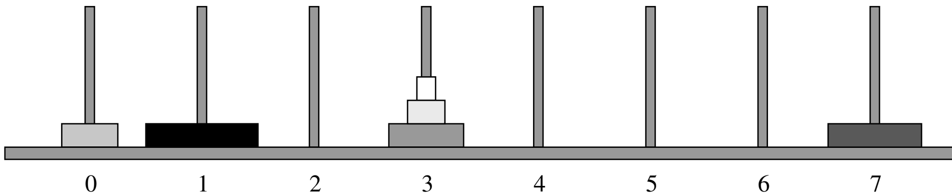

Figure 7 State corresponding to vertex labeled **173033** in H_8^6

We list the labels of its twenty-two adjacent vertices in a table.

Disk	to peg 0	to peg 1	to peg 2	to peg 3	to peg 4	to peg 5	to peg 6	to peg 7
1	**173030**	**173031**	**173032**		**173034**	**173035**	**173036**	**173037**
2								
3		**173133**	**173233**		**173433**	**173533**	**173633**	**173733**
4								
5		**113033**	**123033**		**143033**	**153033**	**163033**	
6			**273033**		**473033**	**573033**	**673033**	

As another example, note that FIGURE 6 corresponds to the edge between vertices labeled **01302** (top) and **01332** (bottom) in H_4^5. Conversely, we can determine the state from its vertex label.

Notice the vertex labeled $s_d \cdots s_2 s_1$ has $k = |\{s_d, \cdots, s_2, s_1\}|$ occupied pegs. For example, the vertex labeled **173033** in H_8^6 has

$$k = |\{1, 7, 3, 0, 3, 3\}| = 4$$

occupied pegs and thus degree $\binom{8}{2} - \binom{8-4}{2}$, which equals 22, as before. The reader can check the degrees in the now-labeled graphs H_3^3 and H_4^2 shown in FIGURES 8 and 9.

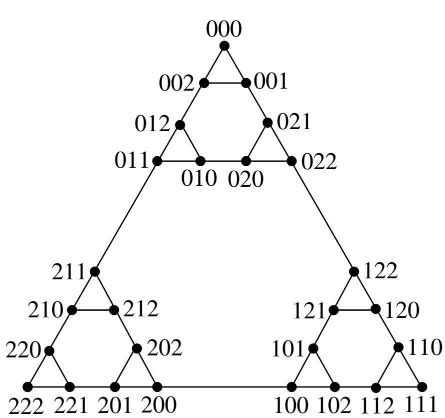

Figure 8 H_3^3 with vertex labels

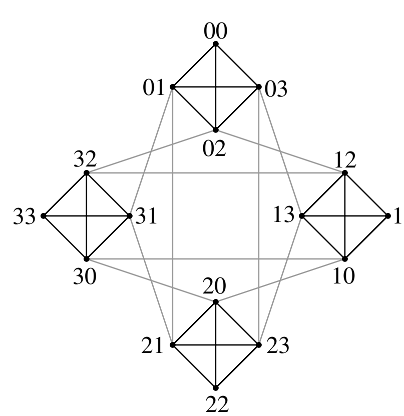

Figure 9 H_4^2 with vertex labels

With this labeling we can now formally define the standard recursive construction of the graphs. We write $v \sim w$ if the vertex labeled v is adjacent to the vertex labeled w. Any vertex of H_p^d has a label of the form av where a is the peg number for the largest disk and v is the label from the vertex in H_p^{d-1} corresponding to the arrangement of the other disks.

When is $av \sim bw$ in H_p^d? There are two possibilities. If we do not move the largest disk, then $a = b$ and, since we must move a smaller disk, $v \sim w$ in H_p^{d-1}. If we move the largest disk while the other disks remain fixed, then $a \neq b$ but $v = w$. In this case there cannot be any other disks on either peg a or peg b or else the largest disk could not move. Thus, in the state corresponding to v, pegs a and b are empty. We abuse the notation slightly by writing $a, b \notin v$ for short.

As an application, we derive a recursive formula for the number of edges in H_p^d for fixed p, which we denote $e_{d,p}$. An edge where we do not move the largest disk has the form $av \sim aw$ for $a \in \{0, 1, \ldots, p-1\}$ and $v \sim w$ in H_p^{d-1}; thus H_p^d has $pe_{d-1,p}$ edges of this type. An edge where we move the largest disk has the form $av \sim bv$ for $a, b \in \{0, 1, \ldots, p-1\}$ and $v \in H_p^{d-1}$ such that $a, b \notin v$. The vertex labeled v can correspond to any of the $(p-2)^{d-1}$ arrangements of the $d-1$ disks on the pegs other than a and b. Thus H_p^d has $\binom{p}{2}(p-2)^{d-1}$ edges of this type. Therefore, $e_{1,p} = \binom{p}{2}$ and for $d \geq 2$,

$$e_{d,p} = pe_{d-1,p} + \binom{p}{2}(p-2)^{d-1}.$$

The reader can check that our previous count satisfies this recursion.

Thus far we have looked at known properties of the Hanoi graphs. We are now ready to prove a new result. The Hanoi graphs are complicated, but thanks to their symmetry and our convenient labeling, they can be easily colored.

For a positive integer c, a graph can be *c-colored* if there is a way to label the vertices with the colors $0, 1, \ldots, c-1$ such that adjacent vertices are different colors. The *chromatic number* of a graph G is the smallest number of colors needed and is denoted $\chi(G)$. For example, $\chi(H_p^1) = \chi(K_p) = p$.

At any vertex of the full graph H_p^d, the subgraph corresponding to moving only the smallest disk is a copy of $H_p^1 \cong K_p$. Thus $\chi(H_p^d) \geq p$.

To see that p colors suffice, color the vertex labeled $s_d \cdots s_2 s_1$ by the sum of its peg numbers modulo p. That is,

$$\phi(s_d \cdots s_2 s_1) = s_d + \cdots + s_2 + s_1 \pmod{p}.$$

To check that ϕ is a p-coloring, observe that the labels of adjacent vertices differ in exactly one place, corresponding to the sole moved disk between the states.

FIGURE 10 shows this coloring of H_4^3 with white (0), light gray (1), dark gray (2), and black (3).

Alternatively, this coloring can be built recursively. Begin with H_p^1 colored by its vertex labeling. For $d \geq 2$, given p copies of H_p^{d-1} each initially p-colored the same, place the number a in front of each vertex label in the ath copy and twist the coloring of each vertex in that copy by adding a modulo p. Formally, write $\psi(v)$ for the color assigned to the vertex labeled v in H_p^{d-1}, so that the twisted coloring on H_p^d is defined by

$$\phi(av) = \psi(v) + a \pmod{p}.$$

The reader can now verify that each type of edge in H_p^d connects vertices of different colors and also that we obtain the same coloring as before.

Notice that, although the number of vertices and number of edges of the Hanoi graphs each grow exponentially in the number of disks, the chromatic number is independent of the number of disks.

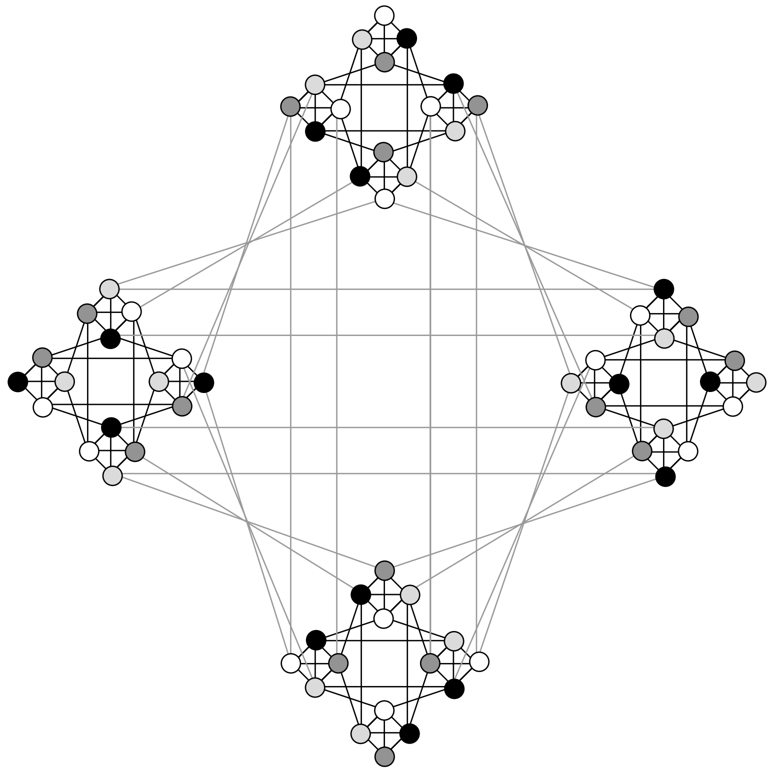

Figure 10 H_4^3 with colored vertices

Another way to measure a graph is by its *independence number*, which is the maximum number of non-adjacent vertices, usually called $\beta(G)$. In the Hanoi graphs, the p^{d-1} vertices of a fixed color in a minimal coloring form an independent set and so $\beta(H_p^d) \geq p^{d-1}$. Conversely, any independent set may include at most one vertex from each copy of K_p corresponding to moving only the smallest disk. As there are p^{d-1} copies, $\beta(H_p^d) = p^{d-1}$.

Further investigation

While we understand much about the Hanoi graphs, there is much we still do not know. Hinz and Parisse have calculated the *chromatic index* (edge-coloring number) of the Hanoi graphs [**8**]. Any permutation of the peg numbers gives an automorphism of the graph. Recently, So Eun Park has shown that these are the only automorphisms of the graph: $Aut(H_p^d) \cong S_p$ [**14**]. Most graph theoretic measures of the Hanoi graphs—including the domination number, covering number, and pebbling numbers—are unknown. Some of these quantities have been calculated for the Sierpiński graphs but not the Hanoi graphs for more than three pegs [**18**].

We are particularly interested in the *diameter*: the maximum over all pairs of vertices of the minimal length of a path connecting them. The minimum number of moves needed to solve the Tower of Hanoi puzzle is bounded by the diameter of the graph and equal to the diameter in the classic 3-peg graph. The diameter of the multipeg graphs are, in general, unknown and it is known that in some cases the diameter is larger than the minimum number of moves. Thus it is not clear whether calculating the diameter is more or less difficult than calculating the minimum number of moves needed to solve the puzzle. Some results on the diameter of variants of the puzzle are known [**1**].

The 3-peg Hanoi graphs are *planar*: they can be drawn in the plane without any edges crossing. Hinz and Parisse [**7**] prove that the only planar Hanoi graphs on more than three pegs are H_4^1 and H_4^2. (We challenge the reader to draw H_4^2 without crossing. If you try and are stuck, consider these possibly cryptic hints: View K_4 as if looking at the top of a tetrahedron and do a little "cat's cradle." In case you are still puzzled, look for a representation of H_4^2 as a planar graph in the October 2010 issue of this MAGAZINE.) For any nonplanar graph, it is natural to ask about the *crossing number*: the minimum number of crossings needed to draw it in the plane. (Technically, a crossing involves only two edges at a time.) Alternatively we might inquire whether there are other surfaces on which the graph can be drawn without crossings; the *genus* of a graph is the smallest genus of such a surface. The genus is no larger than the crossing number, as one can add a bypass handle at each edge crossing, but efficiencies often lead to a smaller genus. The genera of the complete graphs are known, but the crossing numbers are not. Results on the crossing numbers of the related Sierpinski graphs are given by Klavžar and Bojan Mohar [**12**]. The genera and crossing numbers of nonplanar multidisk Hanoi graphs are unknown.

We offer one final direction for further investigation. Poole lists numerous variants of the puzzle [**15**]. For example, in "Straightline Hanoi" on three pegs, we may only move disks to and from the first peg. In "Cyclic Hanoi" the pegs are arranged in a circle and we may only move disks counterclockwise. In "Rainbow Hanoi" the disks are colored and various restrictions are placed on moves based on the color of the disks. In "Multidisk Hanoi" there are multiple copies of each disk (either distinguishable or not). Hinz claims that Lucas suggested the variation of allowing the disks to be out of order at the start—larger disks on smaller ones—subject to the usual rules later in the play. Still other variants allow a larger disk to sit on the next smallest disk, but not any smaller disks than that. To our knowledge, very little about their graphs is known.

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Summary The Tower of Hanoi graphs make up a beautifully intricate and highly symmetric family of graphs that show moves in the Tower of Hanoi puzzle played on three or more pegs. Although the size and order of these graphs grow exponentially large as a function of the number of pegs, p, and disks, d (there are p^d vertices and even more edges), their chromatic number remains remarkably simple. The interplay between the puzzles and the graphs provides fertile ground for counts, alternative counts, and still more alternative counts.

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NOTES

When Is n^2 a Sum of k Squares?

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The square 169 can be written as a sum of two squares $5^2 + 12^2$, as a sum of three squares $3^2 + 4^2 + 12^2$, as a sum of four squares $1^2 + 2^2 + 8^2 + 10^2$, as a sum of five squares $1^2 + 2^2 + 2^2 + 4^2 + 12^2$, and so on for quite a long while. In fact, Jackson, et al. [**5**] note that 169 can be written as a sum of k positive squares for all k from 1 to 155 and first fails as a sum of length 156. The authors go on to ask whether there is any limit to such a string of sums. Specifically, for every positive integer b is there an integer n which can be written as a sum of k positive squares for all k from 1 to b? We assemble a collection of results, most of which have been known for quite some time, to answer this question and, in fact, to specify all possible lengths for sums of squares equal to a given square.

This investigation began when I read a manuscript in which the author proved that a certain combinatorially defined integer $c(k)$ could be written as a sum of k positive integer squares. Although the proof technique was interesting, I wondered if it wouldn't be more surprising to find that a sufficiently large integer *couldn't* be written as a sum of k squares. For that reason, in what follows we address the possible lengths for sums of squares equal to a given integer which may or may not be a square.

Sums of 5 or more positive squares Dickson [**1**] credits Dubouis with publishing the following theorem in 1911. An integer $n \geq 34$ can be written as a sum of k positive squares for all k satisfying $5 \leq k \leq n$ except for $k = n - 13$, $n - 10$, $n - 7$, $n - 5$, $n - 4$, $n - 2$, $n - 1$. Writing 20 years later, Pall [**7**] laments over having duplicated Dubouis' work before noticing the report of it but resists presenting his own proof. Writing over 75 years later still, I suspect that both Dubouis' and Pall's proofs resembled the following.

First we show that no integer n can be written as a sum of k positive squares for $k \in \{n - 13, n - 10, n - 7, n - 5, n - 4, n - 2, n - 1\}$. To see this note that the sum of k positive squares $n = s_1^2 + \cdots + s_k^2$ can be obtained from the sum of n ones by repeatedly replacing s_i^2 of the ones with the single square s_i^2. This replacement reduces the number of summands by $s_i^2 - 1$. For example, replacing four ones, $1 + 1 + 1 + 1$, with a single square 2^2 reduces the number of summands by 3. A replacement of 3^2 ones reduces the number of summands by 8 and larger squares reduce the number of summands by at least 15. A quick check shows that the count of n summands in the sum of all ones cannot be reduced by any of the amounts 1, 2, 4, 5, 7, 10, 13 using reductions of 3 and 8.

We now use induction to show that n can be written as sums of the specified lengths, securing the base case of $n = 34$ with a hand check. For $n > 34$ we add 1^2 to each of

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the sums of squares equal to $n-1$ given by the induction hypothesis. This gives all of the required lengths of sums for n except for a length 5 sum.

The proof is completed by showing that all $n > 34$ can be written as a sum of 5 positive squares. A computer check (an additional hand check for Pall and Dubouis) verifies this for $34 < n \leq 169$. For $n > 169$ we use Lagrange's theorem, which states that every positive integer can be written as a sum of four or fewer positive squares. For $n > 169$, use Lagrange's theorem to write $n - 169$ as a sum of 1, 2, 3 or 4 positive squares. Then add the appropriate representation of 169 as the sum of 4, 3, 2, or 1 positive squares to obtain five positive squares summing to n.

So, except for lengths of 2, 3, and 4, this result specifies all possible lengths for sums of squares equal to a given square. In addition the result greatly simplifies the question in Jackson, et al., since if a square n can be written as a sum of 2, 3, and 4 positive squares then n can be written as a sum of k positive squares for all $1 \leq k \leq n - 14$.

Sums of two positive squares There seems to be some disagreement about when an integer can be written as a sum of two positive squares. In the 1959 article [**3**] the condition is stated that the integer must have the form $4^a n_1 n_2^2$, with integral $a \geq 0$, $n_1 > 1$, the prime factors of n_1 congruent to 1 mod 4 and the prime factors of n_2 congruent to 3 mod 4. In the 2006 book [**6**] the condition is the same except that 4^a is replaced with 2^e, with e a nonnegative integer. In both sources the claims are said to follow easily from previous results, but proofs are not given. However, neither of these conditions include $18 = 2 \times 3^2 = 3^2 + 3^2$ since 18 has no $4k+1$ prime factor. More generally the conditions exclude the numbers $n = m^2 + m^2$ where m has no $4k+1$ prime. Perhaps the authors meant to describe conditions in which n could be written as a sum of two distinct positive squares.

In any case, the correct statement is that a positive integer n can be written as the sum of two positive squares if and only if either n is twice a square or n has at least one $4k+1$ prime factor and all of its $4k+3$ prime factors appear to even powers.

This fact follows easily from the much deeper theory for computing $r_k(n)$ which is defined to be the number of ways of writing n as a sum of k integer squares. In computing $r_k(n)$ the squares of both positive and negative integers as well as 0^2 are allowed and permutations of addends are counted as distinct sums. So, for example $r_2(9) = 4$ since $9 = 0^2 + (\pm 3)^2 = (\pm 3)^2 + 0^2$ are the four ways to express 9 as the sum of two integer squares.

Let $n = 2^k \prod p_i^{a_i} \prod q_j^{b_j}$ be the prime factorization of n with the p_i and q_j being the primes congruent to 1 and 3 mod 4, respectively. Gauss showed that if any of the b_j are odd then $r_2(n) = 0$ and otherwise $r_2(n) = 4\prod(1 + a_i)$. So for example, since $n = 9$ has no $4k+3$ primes to an odd power, and all $4k+1$ primes occur to the zero power, $r_2(9) = 4(1+0) = 4$ as counted above.

Now assume that $n = a^2 + b^2$ is the sum of two positive squares. Either n is twice a square or $a \neq b$ in which case $n = (\pm a)^2 + (\pm b)^2 = (\pm b)^2 + (\pm a)^2$ shows that $r_2(n) \geq 8$. From this it follows that all $4k+3$ primes appear to even powers and there is at least one $4k+1$ prime. Conversely, if $n = 2k^2$, then clearly n is the sum of two nonzero squares. If, on the other hand, all $4k+3$ primes appear to even power and there is at least one $4k+1$ prime, then $r_2(n) \geq 8$. Since at most 4 of these sums can use 0^2, there must be a sum with two positive squares.

Sums of three positive squares When an integer can be written as a sum of three positive squares has not quite been pinned down. Legendre showed that numbers of the form $4^h(8k+7)$ are those which cannot be written as the sum of three *or fewer* positive squares. But this left open the set of numbers which cannot be written as a sum of three positive squares but can be written as a sum of one or two. In 1959 Grosswald, et al., [**3**] proved that there exists a finite set of integers S such that n is not the sum of three positive squares if and only if $n = 4^h q$ where $q = 7 \bmod 8$ or q is an element of the finite set S. They conjectured that $S = \{1, 2, 5, 10, 13, 25, 37, 58, 85, 130\}$ but their proof showed only that the set S is finite.

Despite this disappointment, it is known which *squares* are sums of three positive squares. Hurwitz [**4**] proved that with the exception of $(2^k)^2$ and $(5 \times 2^k)^2$, every positive square can be written as a sum of three positive squares. Fraser and Gordon later gave an elementary proof of this fact in [**2**].

As a digression, note that Hurwitz's result shows that the set S contains no squares other than 1 and 25. So, in considering whether there might be additional numbers in S, we need only consider nonsquares. If n is not a square, then for $n = a^2 + b^2$ neither a nor b are zero and so the orderings in the three sums $0^2 + a^2 + b^2$, $a^2 + 0^2 + b^2$, $a^2 + b^2 + 0^2$ are distinct. If n cannot be written as a sum of three positive squares, then all sums of three squares equal to n must have one of these three forms. Thus if n is not a square, then n cannot be written as a sum of three positive squares if and only if $r_3(n) = 3r_2(n)$. In three hours, a laptop search using Mathematica's built-in SquaresR function verified that the conjectured values for S are correct for $n \leq 5 \times 10^6$.

Sums of four positive squares In [**6**], Pall is credited with showing that n can be written as a sum of four positive squares if and only if n is not one of $\{1, 3, 5, 9, 11, 17, 29, 41\}$ or of the form 2×4^k, 6×4^k, 14×4^k. In a footnote of the cited work [**7**], Pall says that "the reader will have no difficulty in proving [this result] by using the following classical result, which was first stated by Fermat, and was first proved by Legendre in 1798. A positive integer is a sum of three [or fewer positive] squares if and only if it is not of the form $4^h(8k + 7)$". With such a challenge I picked up my pen and searched for the proof. Minutes ticked away to hours with my ego sinking all the while. I eventually did hit upon the following proof similar to the one I later found in [**8**].

First note that $4^h(8k + 7) = 0, 4, 7 \bmod 8$. If $n = 2, 3, 4, 6, 7 \bmod 8$, then $n - 13^2 = 1, 2, 3, 5, 6 \bmod 8$ and so $n - 13^2$ is not of the form $4^h(8k + 7)$. Thus for $n > 13^2$, Legendre's results shows that $n - 13^2$ can be written as a sum of three or fewer positive squares. Augment this sum with the appropriate choice from among $13^2 = 5^2 + 12^2 = 3^2 + 4^2 + 12^2$ to obtain four positive squares summing to n. For $n \leq 13^2$, a computer check finds that $\{2, 3, 6, 11, 14\}$ are the only integers in these congruence classes which cannot be written as a sum of four positive squares.

If $n = 1, 5 \bmod 8$, then $n - 26^2 = 5, 1 \bmod 8$. So for $n > 26^2$, $n - 26^2$ can be written as a sum of three positive squares. Augment this sum with the appropriate choice from among $26^2 = 10^2 + 24^2 = 6^2 + 8^2 + 24^2$ to obtain four positive squares summing to n. For $n \leq 26^2$, a computer check finds that $\{1, 5, 9, 17, 29, 41\}$ are the only integers in these congruence classes which cannot be written as a sum of four positive squares.

If $n = 0 \bmod 8$, consideration mod 8 shows that n is a sum of four positive squares if and only if $n/4$ is. Repeated applications of this observation allows n to be written as $4^a 2j$ where $2j \neq 0 \bmod 8$ and n is a sum of four positive squares if and only if $2j$ is. Previous cases show that $2j \neq 0 \bmod 8$ is not a sum of four positive squares only for $2j = 2, 6, 14$.

Conclusion What, then, are the possible lengths for sums of squares equal to a given positive square?

The possible lengths of 5 and higher are specified by Dubouis' result for squares 36 and above. A direct check shows that the same result holds for 16 and 25 and that the possible sum lengths for 9 are 1, 3, 6, 9. Since a square cannot also be twice a square, the squares which can be written as a sum of two positive squares are those with a prime factor congruent to 1 mod 4. We see that among positive squares, $(2^k)^2$ and $(5 \times 2^k)^2$ are the only ones which cannot be written as a sum of three positive squares and that 1 and 9 are the only ones which cannot be written as a sum of four positive squares.

Combining these conditions, we learn that with the exception of $(5 \times 2^k)^2$, a square can be written as sums of 2, 3, and 4 positive squares if and only if it has at least one prime factor congruent to 1 mod 4. Moreover such a square n can be written as a sum of k positive squares for all k from 1 to $n - 14$.

The first few squares meeting the combined conditions are 169, 225, 289, 625, 676, 841, 900. Going out a little farther we find $n = 1\,000\,002\,000\,001 = (101 \times 9901)^2$ with 101 being a prime congruent to 1 mod 4. So this square can be written as a sum of k positive squares for all k from 1 to 1 000 001 999 987, making 169's run of 155 look not so special after all.

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Summary This note shows that with the exception of $(5 \times 2^k)^2$, an integer square can be written as sums of 2, 3, and 4 positive squares if and only if it has at least one prime factor congruent to 1 mod 4. Moreover such a square n can be written as a sum of k positive squares for all k from 1 to $n - 14$. The question of when a non-square can be written as a sum of k positive squares is also examined.

How Fast Will We Lose?

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Two players X and Y play a gambling game. They start with bankrolls of x and y dollars respectively, where x and y are positive integers and $(x, y) \neq (1, 1)$. They repeatedly flip a coin, which may be a fair or unfair coin. When heads appears, X wins and receives one dollar from Y; when tails appears, X loses and pays one dollar to Y.

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The game continues until one player runs out of money. Let L be the event that X loses the match; that is, that it is X who ends the game with a zero balance.

We assume that the flips are independent. We write p for the probability that X wins a given flip, and we always write q for $1 - p$. Then the probability that X loses is

$$\mathbf{Pr}(L) = q^x \frac{p^y - q^y}{p^{x+y} - q^{x+y}} \quad (p \neq q); \quad \mathbf{Pr}(L) = \frac{y}{x+y} \quad \left(p = q = \frac{1}{2}\right). \tag{1}$$

This is a well-known formula. Our gambling game is called "gambler's ruin," and can also be described as a random walk on the integers with two absorbing barriers. A classical reference is Feller [**1**], chapters III and XIV; see especially equations (3.4) and (3.5) in section XIV.3. The theory goes back over 300 years, and early investigators include Huygens, DeMoivre, Monmart, and two Bernoullis. A good source, both for history and results, is Takács [**4**]. Formula (1) is also used in [**3**], for which this paper is a sequel.

In this paper, we study the probability of the event L_n that X loses in *exactly* n flips. DeMoivre calculated this probability in 1718, but his formula was quite complicated; see [**4**], equations (13) and (12). Our goal is to give a simple method for finding these probabilities. As explained in the last section of [**3**], this will involve the parallel goal of counting the number $c_n = c_n(x, y)$ of different sequences of H and T of length n that lead to losing in exactly n flips. Also, given that X loses, we determine the expected time it will take to lose.

For X to go broke, X must lose x more coin flips than X wins. Thus, for some integer $k \geq 0$, the sequence consists of $x + k$ tails and k heads. The probability of each such sequence is $q^{x+k}p^k$, and the number of such sequences is c_{x+2k}. Thus $\mathbf{Pr}(L_{x+2k}) = c_{x+2k}q^{x+k}p^k$. If n is not of the form $x + 2k$, then $\mathbf{Pr}(L_n) = 0$. Therefore

$$\mathbf{Pr}(L) = \sum_{k=0}^{\infty} c_{x+2k} q^{x+k} p^k. \tag{2}$$

As noted in [**3**], the numbers c_{x+2k} are the coefficients for the power series of a certain rational function $g = g_{x,y}$. This means that g is a *generating function* for the sequence $\{c_{x+2k}\}$, $k = 0, 1, 2, \ldots$.

First we rewrite equation (1) using $S_n = p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1}$, which is positive for $0 \leq p \leq 1$. Observe that

$$S_n = \frac{p^n - q^n}{p - q} \quad (p \neq q) \quad \text{and} \quad S_n = \frac{n}{2^{n-1}} \quad \left(p = q = \frac{1}{2}\right). \tag{3}$$

It follows that

$$\mathbf{Pr}(L) = q^x \frac{S_y}{S_{x+y}} \quad \text{for} \quad 0 \leq p \leq 1; \tag{4}$$

to see this, for $p \neq q$ divide the numerator and denominator in (1) by $p - q$, and for $p = \frac{1}{2}$, note that

$$q^x \frac{S_y}{S_{x+y}} = \left(\frac{1}{2}\right)^x \frac{y}{2^{y-1}} \cdot \frac{2^{x+y-1}}{x+y} = \frac{y}{x+y}.$$

LEMMA. *The expression S_n may be expressed as a polynomial in $u = pq$ with integer coefficients.*

Proof. For $n = 1$ or $n = 2$, (3) reduces to 1 so that $S_1 = S_2 = 1$. Since

$$p^{n+1} - q^{n+1} = p^n p - q^n q = p^n(1-q) - q^n(1-p)$$
$$= (p^n - q^n) - pq(p^{n-1} - q^{n-1}),$$

for $p \neq q$ and $n \geq 2$ we have from (3) that

$$S_{n+1} = S_n - pqS_{n-1} = S_n - uS_{n-1}. \tag{5}$$

This identity also holds for $p = \frac{1}{2}$, which can be verified directly or by using a continuity argument. The lemma follows by induction. ∎

Iterating (5), we obtain the sample calculations summarized in TABLE 1.

TABLE 1

S_3	S_4	S_5	S_6
$1-u$	$1-2u$	$1-3u+u^2$	$1-4u+3u^2$

Set the expressions in (2) and (4) for $\mathbf{Pr}(L)$ equal and cancel q^x from both sides of the resulting identity. Setting $u = pq$, we obtain the identity

$$\sum_{k=0}^{\infty} c_{x+2k}u^k = \frac{S_y}{S_{x+y}}. \tag{6}$$

We write $g(u) = g_{x,y}(u)$ for the rational function $\frac{S_y}{S_{x+y}}$. From (6) and (4), we have

$$g(u) = \sum_{k=0}^{\infty} c_{x+2k}u^k \quad \text{and} \quad \mathbf{Pr}(L) = q^x g(u). \tag{7}$$

We call g the loss function of X for the parameters x and y (in the variable u), and we call the coefficients of the Maclaurin expansion in (7) the loss sequence of X for these parameters. To repeat, the first term in the loss sequence is always $c_x = 1$.

THEOREM. *Given the loss sequence c_{x+2k}, we have*

$$\mathbf{Pr}(L_{x+2k}) = c_{x+2k}q^{x+k}p^k \quad \textit{for integers} \quad k \geq 0. \tag{8}$$

Similarly, there is a win sequence d_{y+2k} for X, based on Y's loss function $g_{y,x}$, so that X's probability of winning in exactly $y + 2k$ steps is $d_{y+2k}p^k q^{y+k}$.

Note that, in the beginning, we had equation (2) but we did not know the coefficients. Equation (7) gets the power series to represent a rational function g. Now by direct means, we can obtain the rational function, then its power series, and then easily read off as many coefficients as we like. This is valid because of the uniqueness theorem for power series: If two power series agree on an interval, then their coefficients are equal.

To illustrate the Theorem, see TABLE 2. For example, from the $(x, y) = (4, 2)$ line, we conclude that $\mathbf{Pr}(L_4) = q^4$, $\mathbf{Pr}(L_6) = 4q^5p$, $\mathbf{Pr}(L_8) = 13q^6p^2$, $\mathbf{Pr}(L_{10}) = 40q^7p^3$, etc. Note also that the number of ways of losing in 22 flips is 29,524.

The loss functions in TABLE 2 were obtained using equation (6) and the results in TABLE 1. Most of the loss sequences in TABLE 2 can be verified by rewriting the

TABLE 2

x	y	Loss Function $g(u)$	Loss Sequence—first ten terms
1	2	$1/(1-u)$	1, 1, 1, 1, 1, 1, 1, 1, 1, 1
1	3	$(1-u)/(1-2u)$	$1, 1, 2, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8$
1	4	$(1-2u)/(1-3u+u^2)$	1, 1, 2, 5, 13, 34, 89, 233, 610, 1597
2	4	$(1-2u)/(1-4u+3u^2)$	1, 2, 5, 14, 41, 122, 365, 1094, 3281, 9842
5	1	$1/(1-4u+3u^2)$	1, 4, 13, 40, 121, 364, 1093, 3280, 9841, 29524
4	2	$1/(1-4u+3u^2)$	1, 4, 13, 40, 121, 364, 1093, 3280, 9841, 29524
3	3	$1/(1-3u)$	$1, 3, 3^2, 3^3, 3^4, 3^5, 3^6, 3^7, 3^8, 3^9$

loss function using partial fractions and then using the expansion $\frac{1}{1-w} = \sum_{k=0}^{\infty} w^k$. For example, for $(x, y) = (1, 3)$, we obtain

$$g(u) = \frac{1-u}{1-2u} = 1 + \frac{u}{1-2u} = 1 + \sum_{k=1}^{\infty} 2^{k-1}u^k,$$

which explains the powers of 2 in the loss sequence. The relationship $g_{2,4}(u) = ug_{4,2}(u) + \frac{1}{1-u}$ explains why the loss sequences in lines (4, 2) and (2, 4) look similar.

The sequence for $(x, y) = (1, 4)$ in TABLE 2 no doubt looks familiar. In fact, it is $1, f_1, f_3, f_5, \ldots$ where f_n is the Fibonacci sequence ($f_1 = f_2 = 1$, $f_3 = 2$, $f_4 = 3$, $f_5 = 5, \ldots$). To see this, we note that

$$f_1 + f_2z + f_3z^2 + f_4z^3 + f_5z^4 + \cdots = \frac{1}{1-z-z^2};$$

see, for example, formulas (6.116) and (6.117) in [**2**]. Also

$$f_1 - f_2z + f_3z^2 - f_4z^3 + f_5z^4 - \cdots = \frac{1}{1+z-z^2}.$$

Adding, we obtain

$$f_1 + f_3z^2 + f_5z^4 + \cdots = \frac{1-z^2}{(1-z^2)^2 - z^2}.$$

Replacing z^2 by u, we get for $0 < u < 1$,

$$f_1 + f_3u + f_5u^2 + \cdots = \frac{1-u}{1-3u+u^2},$$

and so

$$1 + f_1u + f_3u^2 + f_5u^3 + \cdots = 1 + u \cdot \frac{1-u}{1-3u+u^2} = \frac{1-2u}{1-3u+u^2}.$$

The rational function on the right is the loss function $g(u)$ in TABLE 2 for $(x, y) = (1, 4)$, and we now see why the corresponding loss sequence consists of Fibonacci numbers.

We return to the power series in (7). $\mathbf{Pr}(L)$ is defined for all p between 0 and 1 inclusive. Hence, from (7), $\sum_{k=0}^{\infty} c_{x+2k}u^k$ converges for $p = \frac{1}{2}$ or $u = \frac{1}{4}$, so the radius of convergence R of the Maclaurin series of any loss function, with $(x, y) \neq (1, 1)$, obeys $\frac{1}{4} \le R < 1$. If we set $u = \frac{1}{4}$ and $p = q = \frac{1}{2}$ in (7), and if we use the second equation of (1), we obtain the following useful relation for the loss sequence:

$$\sum_{k=0}^{\infty} c_{x+2k} \left(\frac{1}{4}\right)^k = g\left(\frac{1}{4}\right) = 2^x \mathbf{Pr}(L) = \frac{2^x y}{x+y}. \tag{9}$$

This shows that, given the value of x and the loss sequence of X, the value of y is uniquely determined. As an example, suppose the loss sequence is one whose general term is 3^k, $k \ge 0$. If $x = 3$, then by using (9), y is determined by the equations $\sum_{k=0}^{\infty} (\frac{3}{4})^k = 4 = \frac{8y}{3+y}$, which has unique solution $y = 3$.

Different pairs (x, y) may yield the same loss function. For example, $(x, y) = (n, 1)$ yields the same loss function as $(x, y) = (n - 1, 2)$. In each case, the common loss function is $1/S_{n+1}$. However, one can never find three distinct pairs (x, y) that have the same loss function. To see this, note that if for $n > 1$, we arrange the powers of u in the expansion of S_n in ascending order as in Table 1, the first two terms in this expansion will be

$$1 - (n-2)u. \tag{10}$$

This is easily proved by induction using the defining relation $S_{n+1} = S_n - uS_{n-1}$. Now suppose that two pairs (x, y) and (x^*, y^*) yield the same loss function, so that $S_y/S_{x+y} = S_{y^*}/S_{x^*+y^*}$ and

$$S_y S_{x^*+y^*} = S_{x+y} S_{y^*}. \tag{11}$$

First suppose that both y and y^* are greater than 1. Performing the multiplications of polynomials in (11) and using (10), we see that the start of the calculation gives

$$[1 - (y-2)u][1 - (x^* + y^* - 2)u] = [1 - (x + y - 2)u][1 - (y^* - 2)u].$$

Equating coefficients of u, we find that $x = x^*$. But then $y = y^*$ by the statement following equation (9).

Now suppose that $y = 1$, so we are investigating the case when $(x, 1)$ and (x^*, y^*) yield the same loss function. Then $S_1/S_{x+1} = S_{y^*}/S_{x^*+y^*}$ and $S_{x^*+y^*} = S_{x+1}S_{y^*}$. The same analysis as in the last paragraph leads to $x^* = x$ if $y^* = 1$, and $x^* = x - 1$ if $y^* > 1$. We are left with the case that $(x, 1)$ and $(x - 1, y^*)$ give the same loss function. For $y^* = 2$ we already observed, prior to equation (10), that this happens. In general, there cannot be three such pairs $(x, 1)$, $(x - 1, y^*)$, $(x - 1, y^{**})$ because, as we noted after equation (9), the y value is uniquely determined by the x value and the loss sequence. Thus only one y value can go with $x - 1$ and $y^* = y^{**}$.

Finally, here are two questions that come to mind.

QUESTION 1. Can an infinite number of the loss functions have a common root?

QUESTION 2. Our main ideas are actually "probability free" in their definition. Can one give, in a manner as simple as ours, a method of determining the loss function for any (x, y) without referring to the probability result (1)?

Average time to lose As promised, we compute the expected time it will take to lose, given that we lose. If T represents the number of flips before losing, then we want the conditional expectation $E(T|L)$ and this equals

$$\frac{1}{\mathbf{Pr}(L)} \sum_{k=0}^{\infty} (x + 2k) c_{x+2k} q^{x+k} p^k = \frac{xq^x g(u) + 2q^x u g'(u)}{q^x g(u)} = x + 2u \frac{g'(u)}{g(u)}.$$

For $(x, y) = (4, 2)$, we have $g'(u)/g(u) = \frac{4-6u}{3u^2-4u+1}$, so the expected number of flips is

$$4 + 2pq \cdot \frac{4 - 6pq}{3p^2q^2 - 4pq + 1}.$$

For $p = q = \frac{1}{2}$ and $u = \frac{1}{4}$, the expected time to lose is 32/3.

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Summary In a version of gambler's ruin, players start with x and y dollars respectively, and flip coins for one dollar per flip until one player runs out of money. This is a random walk with two absorbing barriers. We consider the number of ways for the first player to lose on the nth flip, for $n = x, x+2, \dots$. We use probabilistic arguments to construct generating functions for these quantities along with explicit methods for computing them. This paper builds on the paper by Hirshon and De Simone, *Mathematics Magazine* **81** (2008) 146–152.

More Polynomial Root Squeezing

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Suppose you're looking at the graph of a polynomial $y = p(x)$ in a java applet, with blue dots on the x-axis indicating the polynomial's roots, and red dots on the x-axis showing the positions of the critical points. Let's assume that all the roots are real and that you grab the blue dots and move them around on the x-axis. As you do this, what happens to the red dots?

This is a fair question because the roots determine the polynomial up to a constant multiple, and they determine the critical points exactly. For simplicity (and without loss of generality) we will only consider monic polynomials (that is, polynomials with leading coefficient 1).

If you move all the blue points (roots) the same amount, the whole graph just translates, and all the red dots simply move along for the ride. If you move all the roots in the same direction but by different amounts, it seems reasonable that the critical points all move in that same direction. This is in fact true, according to the Polynomial Root Dragging Theorem (see [**1**], [**3**]). But suppose you take two roots and symmetrically squeeze them closer to each other, something we call polynomial root squeezing. Then

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what do the critical points do? In [**2**], Boelkins, From and Kolins answer this for critical points that are outside the interval between the two selected roots. In this article we extend their analysis to cover critical points at or between the two squeezed roots.

Notation and definitions Let $p(x)$ be a monic degree-n polynomial with real roots $r_1 \le r_2 \le \cdots \le r_n$ and critical points $c_1 \le c_2 \le \cdots \le c_{n-1}$. Rolle's Theorem tells us that there is a critical point strictly between each pair of adjacent roots. We know that wherever there are r roots together at a single point, there are also $(r-1)$ critical points. So we have

$$r_1 \le c_1 \le r_2 \le c_2 \le \cdots \le c_{n-1} \le r_n \tag{1}$$

with $r_i < c_i < r_{i+1}$ whenever $r_i < r_{i+1}$. By polynomial root squeezing we mean selecting two indices i and j with r_i strictly less than r_j; we then move the smaller root from r_i to $r_i + d$ and the larger root from r_j to $r_j - d$, where $d > 0$. We insist that $d < \frac{r_i + r_j}{2}$, so that the roots don't pass each other.

As an example, consider the polynomial $p(x) = x^2(x+1)(x-2)$. It has single roots at -1 and 2, and a double root at 0. Its critical points are at (approximately) $-.693$, 0, and 1.443. After squeezing the roots at -1 and 2 to $-.5$ and 1.5 respectively, the polynomial becomes $\tilde{p}(x) = x^2(x+.5)(x-1.5)$. The left critical point moves to the right from $-.693$ to $-.343$, and the right critical point moves to the left from 1.443 to 1.093. However the center critical point remains at zero. This example is illustrated in FIGURE 1.

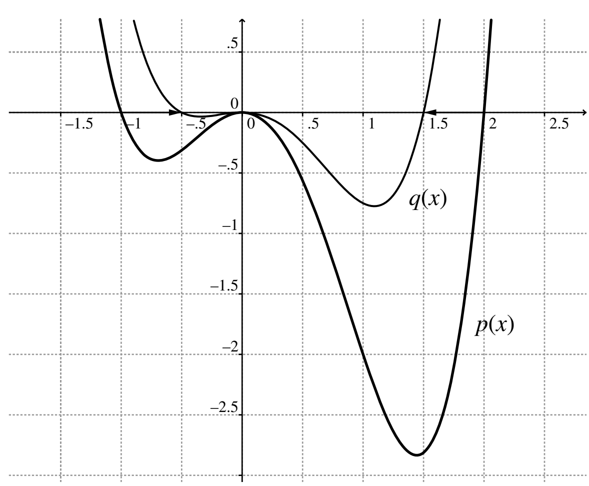

Figure 1 Two roots of the polynomial $p(x) = x^2(x+1)(x-2)$ have been squeezed together to form $\tilde{p}(x)$. In this example, $x = 0$ is a critical point of $p(x)$ and $q(x)$.

Why doesn't the critical point at zero move? It is because $x = 0$ is a repeated root of $\frac{p(x)}{(x+1)(x-2)}$, and as long as this repeated root remains fixed, so must the critical point. More generally, if c_k is a repeated root of $\frac{p(x)}{(x-r_i)(x-r_j)}$, then c_k will remain a critical point when r_i and r_j are squeezed together. For this reason, we say that a critical point is **stubborn** if it is a repeated root of $\frac{p(x)}{(x-r_i)(x-r_j)}$, and **ordinary** otherwise.

A stubborn critical point can move if it lies at r_i or r_j. If r_i (or r_j) lies at a repeated root of multiplicity greater than two, then there is a repeated stubborn critical point there. When r_i is dragged to the right, one of the stubborn critical points will move to the right, while the others will remain fixed. In order to state the theorem as succinctly as possible we exclude the case of stubborn critical points and leave the details as an exercise.

The theorem Boelkins, From and Kolins [**2**] proved the Polynomial Root Squeezing Theorem. That theorem explains how squeezing two roots together affects the critical points that are outside of the interval between the two squeezed roots. Our proof of the Polynomial Root Squeezing Theorem extends their analysis to the critical points that lie at or between the two squeezed roots.

THEOREM. *If the roots at r_i and r_j move equal distances toward each other, then each ordinary critical point moves toward $(r_i + r_j)/2$. If the roots at r_i and r_j move equal distances away from each other, then each ordinary critical point moves away from $(r_i + r_j)/2$.*

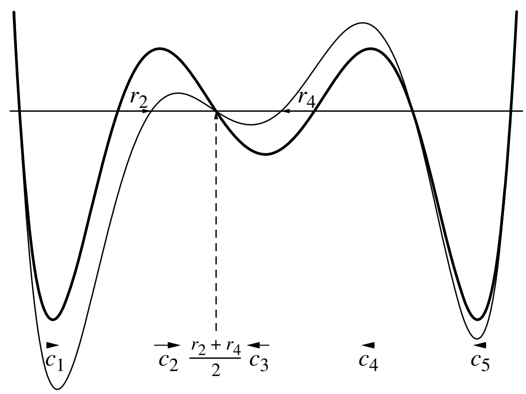

Figure 2 The Polynomial Root Squeezing Theorem: when we drag r_2 and r_4 together, the critical points move toward $(r_2 + r_4)/2$.

Proof. We prove the root squeezing part of the theorem. The root separating part (moving r_i and r_j equal distances away from each other) follows similarly.

Let $p(x)$ be a polynomial of degree n with (possibly repeated) real roots $r_1 \leq r_2 \leq \cdots \leq r_n$, $r_i < r_j$ and c_k any critical point of $p(x)$. Let $\tilde{p}(x)$ be the polynomial that results from squeezing r_i and r_j a fixed distance d, with $0 \leq d < \frac{1}{2}\left(r_j - r_i\right)$. That is

$$\begin{aligned}\tilde{p}(x) &= (x - r_i - d)(x - r_j + d)\prod_{k\neq i,j}(x - r_k)\\ &= (x - r_i - d)(x - r_j + d)q(x).\end{aligned}$$

Denote the roots of $\tilde{p}(x)$ by $\tilde{r}_1 \leq \tilde{r}_2 \leq \ldots \leq \tilde{r}_n$ and the critical points by $\tilde{c}_1 \leq \tilde{c}_2 \leq \ldots \leq \tilde{c}_{n-1}$.

If c_k lies outside the interval from r_i to r_j, then the conclusion follows from [**2**]. (It also follows from a slight variation of the reasoning below.) If c_k is between r_i and $r_i + d$, or between $r_j - d$ and r_j (that is, if one of the moving roots passes by c_k) then the result follows from counting intervals in (1).

We now assume that c_k is not at a repeated root of p and that $r_i + d < c_k < r_j - d$. Our goal is to compare c_k and $\tilde{c_k}$. We do so by investigating $\tilde{p}'(c_k)$. Let

$$p(x) = (x - r_i)(x - r_j)q(x),$$

so that

$$p'(x) = (x - r_i + x - r_j)q(x) + (x - r_i)(x - r_j)q'(x), \tag{2}$$

and

$$\tilde{p}'(x) = (x - r_i + x - r_j)q(x) + (x - r_i - d)(x - r_j + d)q'(x). \quad (3)$$

Subtracting (2) from (3) yields

$$\tilde{p}'(c_k) = d(r_j - r_i - d)q'(c_k). \quad (4)$$

Since $r_j - r_i - d > 0$, this implies that $\tilde{p}'(c_k)$ and $q'(c_k)$ have the same sign.

Without loss of generality we assume that $p(x) < 0$ on (r_k, r_{k+1}) and that $|c_k - r_i| < |c_k - r_j|$ (The cases where $|c_k - r_i| > |c_k - r_j|$ and or $p(x) > 0$ are similar.) Since $r_i < c_k < r_j$, it follows that $(c_k - r_i)(c_k - r_j) < 0$ so that $q(c_k) > 0$. As $p'(c_k) = 0$,

$$0 = p'(c_k) = (c_k - r_i + c_k - r_j)q(c_k) + (c_k - r_i)(c_k - r_j)q'(c_k).$$

An analysis of the sign of the terms, with the assumption that $|c_k - r_i| < |c_k - r_j|$, implies that $q'(c_k) < 0$. It then follows from (4) that $\tilde{p}'(c_k) < 0$.

Since $p(c_k) < 0$, the equation

$$p(c_k)(c_k - r_i - d)(c_k - r_j + d) = \tilde{p}(c_k)(c_k - r_i)(c_k - r_j)$$

implies that $\tilde{p}(c_k) < 0$. Since we assume that $r_i + d < c_k < r_j - d$ and c_k is not a repeated root of p, it follows that $\tilde{r}_k = r_k$ or $\tilde{r}_k = r_i + d$ while $\tilde{r}_{k+1} = r_{k+1}$ or $\tilde{r}_{k+1} = r_j - d$. In all four cases, $\tilde{r}_k < c_k < \tilde{r}_{k+1}$ with $\tilde{p}(c_k) < 0$ which implies that $\tilde{p}(x) < 0$ on $(\tilde{r}_k, \tilde{r}_{k+1})$. Therefore $\tilde{p}'(x)$ changes sign from negative to positive at $\tilde{c}_k$. As $\tilde{p}'(c_k) < 0$, it follows that $c_k < \tilde{c}_k$ and $\tilde{c}_k$ has moved toward $(r_i + r_j)/2$. ■

This extended version of the Polynomial Root Squeezing Theorem completely characterizes the behavior of all the critical points when distinct roots are squeezed or separated a uniform distance. In every case, if a critical point moves at all, it moves in the same direction as the moving root that is nearest to it.

Unfortunately, this intuition does not help us when two distinct roots are squeezed together a nonuniform distance. Neither does it tell us what happens when more than two roots are moved simultaneously. These problems could prompt some interesting undergraduate research.

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Summary Given a polynomial with all real roots, the Polynomial Root Dragging Theorem states that moving one or more roots of the polynomial to the right will cause every critical point to move to the right, or stay fixed. But what happens to the position of a critical point when roots are dragged in opposite directions? In this note we discuss the Polynomial Root Squeezing Theorem, which states that moving two roots, r_i and r_j, an equal distance toward each other without passing other roots, will cause each critical point to move toward $(r_i + r_j)/2$, or remain fixed.

A Counterexample to Integration by Parts

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The integration-by-parts formula

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$$

carries with it an implicit quantification over functions f, g to which the formula applies. So, what conditions must f and g satisfy in order for us to be able to apply the formula?

A natural guess—which some teachers might even offer to a student who raised the question—would be that this formula applies whenever f and g are differentiable. Clearly this condition is necessary, since otherwise the integrands $f'(x)g(x)$ and $f(x)g'(x)$ are not defined. But is this condition sufficient? We will show that it is not. That is, we will give an example of two differentiable functions f, g on $[0, 1]$ for which the definite integrals $\int_0^1 f'(x)g(x)\,dx$ and $\int_0^1 f(x)g'(x)\,dx$ do not exist (the former is $-\infty$ and the latter is $+\infty$); it follows that the functions $f'(x)g(x)$ and $f(x)g'(x)$ do not have antiderivatives on the interval $[0, 1]$, so that the indefinite integrals $\int f'(x)g(x)\,dx$ and $\int f(x)g'(x)\,dx$ do not exist.

A cautious teacher might instead reply that the theorem holds whenever f and g are differentiable and $f'g$ and fg' are integrable. While this version of the theorem is true, it cannot be applied in cases where one does not know ahead of time that the integral one is trying to compute actually exists. One wants an integration-by-parts theorem that includes the integrability of $f'(x)g(x)$ as part of its conclusion, not as part of its hypothesis.

Before we give our counterexample to the naive interpretation of the integration by parts formula, or state what we think the teacher should say, we point out that the formula holds if either f' or g' is continuous. For instance, if f' is continuous, then (since g is continuous) the product $f'g$ is continuous; but then the function $f'g$ must have an antiderivative h, and consequently the function fg' must have an antiderivative too, namely $fg - h$. So any counterexample to the naive interpretation of integration by parts must feature differentiable functions f, g whose derivatives are not continuous, such as the famous function $x^2 \sin 1/x$ (extended to a function on all of $\mathbf{R}$ by continuity) and its relatives. Moreover, it will not do to let f and g be the same function of this sort, since the function ff' always has an antiderivative, namely $\frac{1}{2}f^2$.

Our counterexample is the pair of functions

$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x^4}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

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and

$$g(x) = \begin{cases} x^2 \cos\left(\dfrac{1}{x^4}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

on the interval [0, 1]. Both functions are continuous on [0, 1] and differentiable on [0, 1]. Indeed, if we consider f and g as defined above to be defined on all of $\mathbf{R}$, both functions are differentiable everywhere; for, away from 0 we can use the chain rule, while at 0 we have $|(f(h) - f(0))/(h - 0)| = |f(h)/h| \leq |h^2/h| = |h|$ so that $f'(0) = \lim_{h \to 0}(f(h) - f(0))/(h - 0) = 0$, and likewise $g'(0) = 0$. Obviously, the integral

$$\int_0^1 [f(x)g(x)]'\, dx$$

exists. However, we will show that both integrals

$$\int_0^1 f'(x)g(x)\, dx \quad \text{and} \quad \int_0^1 f(x)g'(x)\, dx$$

are divergent. It suffices to show that the first integral is divergent. For $x \neq 0$,

$$f'(x) = 2x \sin\left(\frac{1}{x^4}\right) - 4x^2 \cos\left(\frac{1}{x^4}\right) \frac{1}{x^5}.$$

The first term in this representation of $f'(x)$ is continuous, and $g(x)$ is continuous, so their product is continuous and therefore integrable. So, we focus on the second term times $g(x)$, namely

$$\begin{aligned} -4 \int_0^1 x^2 \cos\left(\frac{1}{x^4}\right) \frac{1}{x^5}\, g(x)\, dx &= -4 \int_0^1 x^4 \cos^2\left(\frac{1}{x^4}\right) \frac{1}{x^5}\, dx \\ &= \int_0^1 x^4 \cos^2\left(\frac{1}{x^4}\right) d\left(\frac{1}{x^4}\right). \end{aligned}$$

After the substitution

$$u = \frac{1}{x^4}$$

the integral turns into

$$-\int_1^\infty \frac{1}{u} \cos^2(u)\, du$$

(with the minus sign coming from the interchange of upper and lower limits of integration). To show that this integral diverges, let k be a positive integer. Then for every u in the interval $[2\pi k - \frac{\pi}{4}, 2\pi k]$ we have

$$\cos^2(u) \geq \frac{1}{2} \quad \text{and} \quad \frac{1}{u} \geq \frac{1}{(2\pi k)}.$$

Therefore,

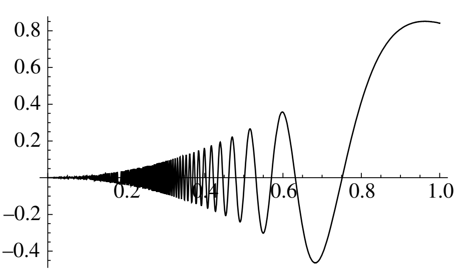
$$\int_1^\infty \frac{1}{u}\cos^2(u)\,du \geq \sum_{k=1}^\infty \int_{2\pi k-\frac{\pi}{4}}^{2\pi k} \frac{1}{u}\cos^2(u)\,du \geq \sum_{k=1}^\infty \frac{1}{(2\pi k)}\frac{1}{2}\frac{\pi}{4} = \frac{1}{16}\sum_{k=1}^\infty \frac{1}{k} = \infty$$

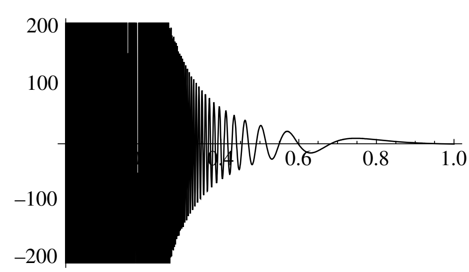
This completes the proof.

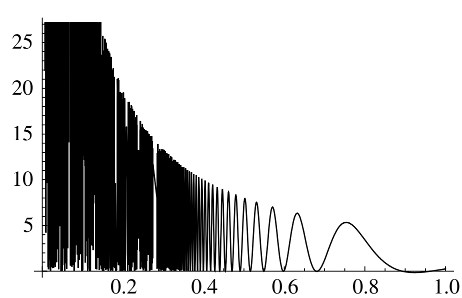
Our analysis shows that the (improper) definite integrals $\int_0^1 f'(x)g(x)\,dx$ and $\int_0^1 f(x)g'(x)\,dx$ do not exist. This in turns shows that the functions $f'(x)g(x)$ and $f(x)g'(x)$ do not have antiderivatives on [0, 1]. For, if these functions had antiderivatives, the fundamental theorem of calculus would yield finite values for the definite integrals.

We have shown that the functions $f'g$ and fg' are not integrable over [0, 1]. It is worth noting that $|f'|$ and $|g'|$ are not integrable over [0, 1] either, as can be shown by a similar method. On the other hand, the function f' is integrable over [0, 1] in the sense that the improper Riemann integral $\int_0^1 f'(x)\,dx$ exists: for all $\epsilon > 0$ the Fundamental Theorem of calculus implies $\int_\epsilon^1 f'(x)\,dx = f(1) - f(\epsilon)$, which converges to $f(1) - f(0)$ as $\epsilon \to 0$, implying that $\int_0^1 f'(x)\,dx$ exists and equals $f(1) - f(0)$. Likewise g' is integrable over [0, 1].

The following three pictures (created with the help of *Mathematica*) illustrate what is going on: they depict the (truncated) graphs of f, f', and $-f'g$ (we show $-f'g$ rather than $f'g$ so that the function will be non-negative rather than non-positive). The continuous function f is integrable, and the discontinuous function f' is integrable because its oscillations balance out, but the non-negative function $-f'g$ is non-integrable.







Some might be inclined to say that our example is actually a vindication of an extended integration by parts theorem that asserts, as important special cases, that if $\int_a^b f'(x)g(x)$ is ∞ then $\int_a^b f(x)g'(x)$ is $-\infty$ and vice versa (and likewise with the signs reversed), and that if either of these integrals "diverges by oscillation" (as in the case for the functions f, g on $[-1, 1]$ given by $x^2\sin(1/x^4)$, $x^2\cos(1/x^4)$ on [0, 1] and $-x^2\sin(1/x^4)$, $-x^2\cos(1/x^4)$ on $[-1, 0]$, respectively) then so does the other. However, to the extent that one might be inclined to treat the integration by parts formula as implicitly asserting that the integrals are well-defined, our example provides a corrective.

Is this corrective needed? We have not found any calculus texts that present a mistaken statement of the integration by parts theorem, but we have found some widely-used web sites that do so (e.g.: "Let u and v be differentiable functions, then $\int uv'\,dx = uv - \int u'v\,dx$"). More common are books and web sites that present the integration by parts formula and give examples without specifying the conditions under which the formula applies. A provocative treatment of other pedagogical aspects of the integration by parts theorem is [**2**].

So, what should the calculus teacher say?

In an ordinary calculus class, the integration by parts formula should be stated as a theorem that begins "If f' and g' are continuous, then ... " (although, as we have noted, it suffices that either f' or g' is continuous).

For a more advanced course (an honors calculus class or an introductory real analysis class), our example could be presented in detail and used to motivate the notion of bounded variation, since the lack of bounded variation of the derivatives of the functions near the origin is the source of the problem. We also mention that, in lieu of adopting the hypothesis that f (or g) is continuously differentiable, one might require that f be Riemann-Stieltjes integrable with respect to dg. Then it can be shown that the integration by parts formula (where the integrals now are Riemann-Stieltjes integrals) is valid, and it is part of the conclusion that g will be Riemann-Stieltjes integrable with respect to df (see [**1**]).

Finally, we mention that if the functions f' and g' are assumed to be integrable in the sense that $\int_0^1 f'(x)\,dx$ and $\int_0^1 g'(x)\,dx$ exist as strict Riemann integrals (and not just as improper Riemann integrals), then the conclusion of the integration by parts theorem applies. Indeed, we only need to know that at least one of f' and g' is Riemann integrable. For, Lebesgue's Theorem states that a (measurable) function is Riemann integrable if and only if it is bounded and its set of discontinuity has Lebesgue measure zero. If g is continuous and f' is Riemann integrable (i.e., it is bounded and its set of discontinuity has Lebesgue measure zero), then so is $f'g$, and the integration by parts theorem applies. Hence it is an essential feature of our counterexample that the functions f' nor g' are not just discontinuous but also non-integrable in the Riemann sense.

Acknowledgment This work was stimulated by conversations with the honors freshman calculus class at UMass Lowell, and also benefited from conversations with Lee Jones of UMass Lowell (who found a different counterexample) and Zbigniew Nitecki of Tufts University.

REFERENCES

1. Tom M. Apostol, *Mathematical Analysis*, 2nd ed., Addison Wesley, Reading, MA, 1974.
2. Jonathan Lewin, "Integration by Parts: Another Example of Voodoo Mathematics," `http://science.kennesaw.edu/~jlewin/fb/integration-by-parts.pdf`.

Summary The authors exhibit two differentiable functions f and g for which the function $f'g$ and fg' are not integrable, so that the integration by parts formula does not apply.

PROBLEMS

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PROPOSALS

To be considered for publication, solutions should be received by November 1, 2010.

1846. *Proposed by Eddie Cheng and Jerrold W. Grossman, Department of Mathematics and Statistics, Oakland University, Rochester, MI.*

For which $n \geq 1$ is it possible to place the numbers $1, 2, \ldots, n$ in some order (a) on a line segment, or (b) on a circle, so that for every s from 1 to $\frac{1}{2}n(n+1)$ there is a connected subset of the segment or circle such that the sum of the numbers on that subset is s?

1847. *Proposed by Panagiote Ligouras, "Leonardo da Vinci" High School, Noci, Italy.*

Let ABC be a scalene triangle. Let h_a, l_a, and m_a be the respective lengths of the height, bisector, and median, of $\triangle ABC$ with respect to A, and let r_a be the exradius of the excircle of $\triangle ABC$ opposite to A. Similarly, define h_b, l_b, m_b, and r_b, with respect to B, and h_c, l_c, m_c, and r_c with respect to C. Prove that

$$\frac{l_a^4(m_a^2 - h_a^2)}{h_a^3 r_a (l_a^2 - h_a^2)} + \frac{l_b^4(m_b^2 - h_b^2)}{h_b^3 r_b (l_b^2 - h_b^2)} + \frac{l_c^4(m_c^2 - h_c^2)}{h_c^3 r_c (l_c^2 - h_c^2)} > \frac{16}{3}.$$

1848. *Proposed by Herb Bailey, Rose–Hulman Institute of Technology, Terre Haute, IN.*

Let N be a base ten positive integer with nonzero last digit. Let N^* be the integer formed by moving the last digit of N to the front. For example, if $N = 867053$ then $N^* = 386705$. Find all N for which N is divisible by N^*.

Math. Mag. **83** (2010) 226–233. doi:10.4169/002557010X494904. © Mathematical Association of America

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Solutions should be written in a style appropriate for this MAGAZINE.

Solutions and new proposals should be mailed to Bernardo M. Ábrego, Problems Editor, Department of Mathematics, California State University, Northridge, 18111 Nordhoff St, Northridge, CA 91330-8313, or mailed electronically (ideally as a LaTeX or pdf file) to mathmagproblems@csun.edu. All communications, written or electronic, should include **on each page** the reader's name, full address, and an e-mail address and/or FAX number.

1849. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Find the sum

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+m}}{(\lfloor\sqrt{n+m}\rfloor)^3},$$

where $\lfloor a\rfloor$ denotes the greatest integer less than or equal to a.

1850. *Proposed by Richard Stephens, Department of Mathematics, Columbus State University, Columbus, GA.*

Let τ be a topology on a finite set X. Define a topology on X to be *regular* if for any nonempty closed $E \subseteq X$ and $x \in X \setminus E$, there exist disjoint open sets U and V in τ such that $E \subseteq V$ and $x \in U$. Prove or disprove that the topological space (X, τ) is regular if and only if τ has a base B which is a partition of X.

Quickies

Answers to the Quickies are on page 232.

Q1001. *Proposed by Herman Roelants, Center for Logic, Institute of Philosophy, University of Leuven, Leuven, Belgium.*

The recursive sequence (a_n) is defined as follows: $a_1 = 0$ and $a_{n+1} = \sqrt{a_n^2+1}+a_n$ for $n \geq 1$. Determine the value of

$$\lim_{n\to\infty}\frac{2^n}{a_n}.$$

Q1002. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.*

Let g be a positive, continuous, real-valued function on $[0, \infty)$, and let

$$f(x) = g(x)\int_0^x \frac{1}{(g(t))^2}\,dt.$$

Prove that f is unbounded on $[0, \infty)$.

Solutions

Locating the intersection of the diagonals **June 2009**

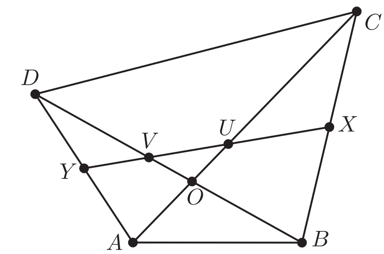
1821. *Proposed by Abdullah Al-Sharif and Mowaffaq Hajja, Yarmouk University, Irbid, Jordan.*

Let $ABCD$ be a convex quadrilateral, let X and Y be the midpoints of sides BC and DA respectively, and let O be the point of intersection of diagonals of $ABCD$. Prove that O lies inside of quadrilateral $ABXY$ if and only if

$$\text{Area}(AOB) < \text{Area}(COD).$$

I. *Solution by Michel Bataille, Rouen, France.*

Let U and V be the points of intersection of $\overline{XY}$ with $\overline{AC}$ and $\overline{BD}$, respectively (see figure).



Let positive real numbers p, q be defined by

$$\overrightarrow{OC} = -p\overrightarrow{OA}, \quad \overrightarrow{OD} = -q\overrightarrow{OB}$$

so that $C = -pA + (1+p)O$ and $D = (1+q)O - qB$.

Then, $2X = B + C = -pA + (1+p)O + B$ and similarly, $2Y = A + (1+q)O - qB$. It follows that the equation of the line XY, in barycentric coordinates (x, y, z) relative to (A, O, B), is

$$(pq + 1 + 2q)x + (pq - 1)y + (pq + 1 + 2p)z = 0,$$

and so the coordinates of U and V are $U = (1 - pq, pq + 1 + 2q, 0)$ and $V = (0, pq + 1 + 2p, 1 - pq)$, that is,

$$2(1+q)\overrightarrow{OU} = (1 - pq)\overrightarrow{OA}, \text{ and } 2(1+p)\overrightarrow{OV} = (1 - pq)\overrightarrow{OB}.$$

Thus O is in the interior of $ABXY$ if and only if $pq > 1$.

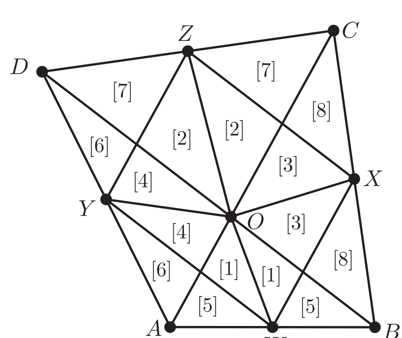
On the other hand,

$$\begin{aligned}\text{Area}(COD) &= \frac{1}{2} OC \cdot OD \cdot \sin(\angle COD) = \frac{1}{2} p\, OA \cdot q\, OB \cdot \sin(\angle AOB) \\ &= pq\,\text{Area}(AOB).\end{aligned}$$

Thus $pq > 1$ if and only if $\text{Area}(COD) > \text{Area}(AOB)$.

II. *Solution by David Getling, Berlin, Germany.*

Let Z and W be the midpoints of $\overline{CD}$ and $\overline{AB}$, respectively. Varignon's Theorem says that $XWYZ$ is a parallelogram. Indeed, $\overline{XW}$ and $\overline{YZ}$ are parallel to $\overline{AC}$ and also $\overline{YW}$ and $\overline{XZ}$ are parallel to $\overline{BD}$. As a consequence O always lies inside this parallelogram. Also, O lies inside $ABXY$ if and only if O lies inside the triangle XYW, that is, O lies inside $ABXY$ if and only if $[WXOY] < [YOXZ]$, where $[WXOY]$ designates the area of $WXOY$.



In the figure, all triangular regions with the same area have been labeled with the same number. The condition $[WXOY] < [YOXZ]$ is equivalent to

$$[1]+[1]+[3]+[4] < [2]+[2]+[3]+[4], \text{ or } [1]+[1]+[3] < [2]+[2]+[3].$$

But $[WBX] = [ABC]/4$, from which $[1]+[3] = [5]+[8]$, and similarly, $[2]+[3] = [7]+[8]$. Thus the condition is equivalent to

$$[1]+[5]+[8] < [2]+[7]+[8], \text{ or } \frac{1}{2}[AOB] = [1]+[5] < [2]+[7] = \frac{1}{2}[COD],$$

which completes the proof.

Also solved by Robert Calcaterra, Robert L. Doucette, Fisher Problem Solving Group, Dmitry Fleischman, Michael Goldenberg and Mark Kaplan, Eugen J. Ionascu, Young Ho Kim (Korea), Omran Kouba (Syria), Victor Y. Kutsenok, Aaron Panchal, Joel Schlosberg, Edward Schmeichel, Marian Tetiva (Romania), and the proposers.

An inequality for $\sqrt[3]{u/v}+\sqrt[3]{v/u}$ **June 2009**

1822. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let u and v be positive real numbers. Prove that

$$\frac{1}{8}\left(17-\frac{2uv}{u^2+v^2}\right) \le \sqrt[3]{\frac{u}{v}}+\sqrt[3]{\frac{v}{u}} \le \sqrt{(u+v)\left(\frac{1}{u}+\frac{1}{v}\right)}.$$

Find conditions under which equality holds.

Solution by Omran Kouba, Damascus, Syria.

We first prove the following inequality. If x is real and $x > 2$, then

$$\frac{1}{8}\left(17-\frac{2}{x(x^2-3)}\right) < x < (x-1)\sqrt{x+2}.$$

Note that

$$8x-17+\frac{2}{x(x^2-3)} = (x-2)\left(8-\frac{(x+1)^2}{x(x^2-3)}\right).$$

Writing $(x+1)^2 = (x^2-3)+2x+4$, and noting that $x^2-3 > 1$ for $x > 2$, it follows that

$$\frac{(x+1)^2}{x(x^2-3)} = \frac{1}{x}+\frac{2x+4}{x(x^2-3)} < \frac{1}{x}+\frac{2x+4}{x} = 2+\frac{5}{x} < 2+\frac{5}{2} = \frac{9}{2}.$$

Hence

$$8x-17+\frac{2}{x(x^2-3)} > (x-2)\left(8-\frac{9}{2}\right) > 0,$$

and the first inequality is proved. To prove the second inequality, note that $x > 2$ implies $\sqrt{x+2} > 2$, and consequently

$$x < 2(x-1) < (x-1)\sqrt{x+2}.$$

For the problem at hand, let $x = \sqrt[3]{u/v}+\sqrt[3]{v/u}$. The Arithmetic Mean–Geometric Mean Inequality implies that $x \ge 2$, with equality if and only if $u = v$. Thus, if $u \ne v$ then $x > 2$, and by the previously proved inequality,

$$\frac{1}{8}\left(17-\frac{2}{x(x^2-3)}\right) < x < (x-1)\sqrt{x+2}.$$

Because $x^3 = u/v + v/u + 3x$, it follows that

$$(x-1)^2(x+2) = x^3 - 3x + 2 = 2 + \frac{u}{v} + \frac{v}{u} = (u+v)\left(\frac{1}{u} + \frac{1}{v}\right)$$

and

$$\frac{2}{x(x^2-3)} = \frac{2}{x^3-3x} = \frac{2}{u/v+v/u} = \frac{2uv}{u^2+v^2}.$$

Therefore

$$\frac{1}{8}\left(17 - \frac{2uv}{u^2+v^2}\right) < \sqrt[3]{\frac{u}{v}} + \sqrt[3]{\frac{v}{u}} < \sqrt{(u+v)\left(\frac{1}{u}+\frac{1}{v}\right)}.$$

Moreover, if $u = v$ then all three expressions in the inequality are equal, so equality holds if and only if $u = v$.

Editor's Note. Stan Wagon verified that the constant $\frac{1}{8}$ in the first inequality cannot be improved. Eugene A. Herman proved the stronger inequality $\frac{4}{9}(5 - uv/(u^2+v^2)) < \sqrt[3]{u/v} + \sqrt[3]{v/u}$. Furthermore, he proved that this is the sharpest possible inequality of the form $a - b(uv/(u^2+v^2)) < \sqrt[3]{u/v} + \sqrt[3]{v/u}$ with $a, b > 0$. Graham Lord generalized in a different vein; he proved that $\frac{1}{8}(17 - 2uv/(u^2+v^2)) < \sqrt[4]{u/v} + \sqrt[4]{v/u}$ and verified that the statement no longer holds with fifth roots.

Also solved by Arkady Alt, Michel Bataille (France), Minh Can, Hongwei Chen, John Christopher, Chip Curtis, Robert L. Doucette, John Ferdinands, Leon Gerber, Michael Goldenberg and Mark Kaplan, Eugene A. Herman, Eugen J. Ionascu and Sarah E. Ewing, Parvis Khalili, Elias Lampakis (Greece), Kee-Wai Lau (China), Graham Lord, José H. Nieto (Venezuela), Northwestern University Math Problem Solving Group, Occidental College Problem Solving Group, Paolo Perfetti (Italy), Gabriel T. Prăjitură, Joel Schlosberg, John L. Simmons (Holland), Nicholas C. Singer, Sanghun Song (Korea), Albert Stadler (Switzerland), David Stone and John Hawkins, Marian Tetiva (Romania), Texas State Problem Solvers Group, Michael Vowe (Switzerland), Stan Wagon, and the proposer. There were two incorrect submissions.

Permutations with k initial entries of the same parity **June 2009**

1823. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.*

Let n and k be positive integers. Find a closed-form expression for the number of permutations of $\{1, 2, \ldots, n\}$ for which the initial k entries have the same parity, but the initial $k+1$ entries do not. (As an example, for the permutation 5712463, the number of initial entries of the same parity is 3, the order of the set $\{5, 7, 1\}$.)

Solution by José H. Nieto, Universidad del Zulia, Maracaibo, Venezuela.

Let $I_n = \{1, 2, \ldots, n\}$. Denote by $E(n,k)$ and $O(n,k)$ the sets of permutations of I_n with just k initial even entries, respectively with just k initial odd entries. The problem asks to find an expression for $p(n,k) = |E(n,k)| + |O(n,k)|$.

If $n = 2m$ is even, the first k entries of a permutation in $E(n,k)$ can be chosen in $m(m-1)\cdots(m-k+1)$ ways, the $(k+1)$th entry in m ways, and the remaining $n-k-1$ entries in $(2m-k-1)!$ ways, hence $|E(2m,k)| = \binom{m}{k}k!m(2m-k-1)!$. By symmetry $|O(2m,k)| = |E(2m,k)|$ and

$$p(2m,k) = 2m\binom{m}{k}k!\,(2m-k-1)!.$$

Analogously, if $n = 2m+1$ then $|E(2m+1,k)| = \binom{m}{k}k!\,(m+1)(2m-k)!$ and $|O(2m+1,k)| = \binom{m+1}{k}k!\,m(2m-k)!$, hence

$$p(2m+1,k) = \left((m+1)\binom{m}{k} + m\binom{m+1}{k}\right)k!(2m-k)!.$$

Both formulas for n even and odd may be resumed as follows:

$$p(n,k) = \left(\left\lceil \frac{n}{2} \right\rceil \binom{\lfloor \frac{n}{2} \rfloor}{k} + \left\lfloor \frac{n}{2} \right\rfloor \binom{\lceil \frac{n}{2} \rceil}{k} \right) k!(n-k-1)!.$$

Editor's Note. Graham Lord observed that if the set I_n is partitioned into sets A and B with $|A| = a$ and $|B| = b$, then the number of permutations of I_n where the first k entries are in A and the next j entries are in B is equal to $\binom{a}{k}k!\,\binom{b}{j}j!\,(n-j-k)!$.

Also solved by Michel Bataille (France), Jany C. Binz (Switzerland), Robert Calcaterra, Chip Curtis, M. N. Deshpande (India), Dmitry Fleischman, Ralph P. Grimaldi, Eugene A. Herman, Peter M. Joyce and Richard F. McCoart Jr., Victor Y. Kutsenok, Elias Lampakis (Greece), Graham Lord, Rob Pratt, Joel Schlosberg, John Sumner and Aida Kadic-Galeb, Nicholas C. Singer, Texas State Problem Solvers Group, Michael Woltermann, and the proposer.

An Intermediate Value Theorem conclusion **June 2009**

1824. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania.*

Let f be a continuous real-valued function defined on [0, 1] and satisfying

$$\int_0^1 f(x)\,dx = \int_0^1 xf(x)\,dx.$$

Prove that there exists a real number c, $0 < c < 1$, such that

$$cf(c) = \int_0^c xf(x)\,dx.$$

Solution by Dave Trautman, Department of Mathematics and Computer Science, The Citadel, Charleston, SC.

Because f is continuous and $\int_0^1 (1-x)f(x)\,dx = 0$, the Mean Value Theorem for Integrals assures the existence of some c_1, $0 < c_1 < 1$, such that $(1-c_1)f(c_1) = 0$. Clearly this means $f(c_1) = 0$. If $\int_0^{c_1} xf(x)\,dx = 0$, then $c = c_1$ proves the required identity. Replacing f by $-f$ if necessary, it can be assumed that $\int_0^{c_1} xf(x)\,dx > 0$. Because the function $G(x) = xf(x)$ is continuous on [0, 1], there exists c_2, $0 \le c_2 < c_1$, such that $G(c_2)$ is the maximum value of G on $[0, c_1]$. For $0 \le x \le c_1$, let

$$H(x) = \int_0^x tf(t)\,dt.$$

Because $c_2 < 1$, it follows that

$$H(c_2) = \int_0^{c_2} tf(t)\,dt \le c_2 G(c_2) < G(c_2).$$

On the other hand,

$$H(c_1) = \int_0^{c_1} tf(t)\,dt > 0 = G(c_1).$$

Thus the Intermediate Value Theorem says that there exists c, $c_2 < c < c_1$, such that $G(c) = H(c)$, that is $cf(c) = \int_0^c xf(x)\,dx$.

Editor's Note. A number of readers pointed out that the same conclusion follows if the hypothesis is replaced by the weaker condition of f being continuous and $f(x_0) = 0$ for some $0 < x_0 < 1$.

Also solved by Michael R. Bacon and Charles K. Cook, Michel Bataille (France), Gerald E. Bilodeau, Michael W. Bosko, Robert Calcaterra, Hongwei Chen, John Christopher, Andrés Fielbaum (Chile), Fisher Problem Solving Group, G.R.A.20 Problem Solving Group (Italy), William Hodge, Eugen J. Ionascu, Parviz Khalili, Elias Lampakis (Greece), Kee-Wai Lau (China), Kim McInturff, Occidental Problem Solving Group, Ángel Plaza and José M. Pacheco (Spain), Edward Schmeichel, Sanghun Song (Korea), Marian Tetiva (Romania), Jeremy Thibodeaux, Thomas P. Turiel, Nicholas J. Willis, and the proposer.

Non-nested subsets of a ring closed under multiplication **June 2009**

1825. *Proposed by Greg Oman and Kevin Schoenecker, The Ohio State University, Columbus, OH.*

Let R be a ring with more than two elements. Prove that there exist subsets S and T of R, both closed under multiplication, and such that $S \not\subseteq T$ and $T \not\subseteq S$. (Note: We do not assume that R is commutative nor do we assume that R has a multiplicative identity.)

Solution by Howard E. Bell, Department of Mathematics, Brock University, St. Catherines, Ontario, Canada.

If R contains an element a such that $a^n \neq 0$ for all $n \in \mathbb{Z}^+$, then the sets $S = \{0\}$ and $T = \{a^n : n \in \mathbb{Z}^+\}$ satisfy the required properties. Assume that R is a nil ring, that is for every $x \in R$ there is a positive integer n such that $x^n = 0$. Let the index of x be the smallest positive integer with this property. If R contains two distinct elements a and b of index 2, then let $S = \{0, a\}$ and $T = \{0, b\}$. Clearly S and T satisfy the required conditions. This case occurs if the maximum index in R is 2. It also occurs when there exists $a \in R$ with index $k \geq 4$, for in this case a^{k-1} and a^{k-2} are two elements of index 2. The only remaining case is that R contains an element a of index 3, in which case a, a^2, and $a + a^2$ are nonzero and $a \neq a^2$, $a \neq a + a^2$, and $a^2 \neq a + a^2$. Thus the sets $S = \{0, a, a^2\}$ and $T = \{0, a + a^2, a^2\}$ satisfy the requirements.

Note. It is possible to insist that $S \cup T$ be commutative, for if R is a noncommutative ring with maximum index 2 and a and b are noncommuting elements of R, then a, b, and $a + b$ all have square zero, so that $ab + ba = 0$ and hence both ab and ba are nonzero. Thus, $S = \{0, ab\}$ and $T = \{0, ba\}$ satisfy the requirements and $S \cup T$ is commutative.

Also solved by Paul Budney, Robert Calcaterra, John Ferdinands, John N. Fitch, Rod Hardy and Alin A. Stancu, Elias Lampakis (Greece), David P. Lang, Missouri State University Problem Solving Group, Justin Neil and Paul Peck, José H. Nieto, Northwestern University Math Problem Solving Group, Éric Pité (France), Gabriel T. Prăjitură, Nicholas C. Singer, John Sumner and Aida Kadic-Galeb, Vadim Ponomarenko, Marian Tetiva (Romania), Texas State University Problem Solvers Group, Gregory P. Wene (Mexico), and the proposers. There was one incorrect submission.

Answers

Solutions to the Quickies from page 227.

A1001. The answer is π. Note that $1/a_2 = 1 = \tan(\pi/2^2)$. By induction, if $1/a_n = \tan(\pi/2^n)$, then for positive angles less than $\pi/2$ the Tangent Half-Angle Formula gives

$$\tan\left(\frac{\pi}{2^{n+1}}\right) = \frac{-1 + \sqrt{1 + \tan^2(\pi/2^n)}}{\tan(\pi/2^n)} = \frac{-1 + \sqrt{1 + a_n^{-2}}}{a_n^{-1}}$$
$$= -a_n + \sqrt{a_n^2 + 1} = \frac{1}{\sqrt{a_n^2 + 1} + a_n} = \frac{1}{a_{n+1}}.$$

Therefore

$$\lim_{n\to\infty} \frac{2^n}{a_n} = \lim_{n\to\infty} \left(2^n \tan\left(\frac{\pi}{2^n}\right)\right) = \pi \lim_{n\to\infty} \left(\frac{2^n}{\pi} \tan\left(\frac{\pi}{2^n}\right)\right) = \pi.$$

A1002. Suppose f is bounded on $[0, \infty)$. Let $h(x) = \int_0^x (g(t))^{-2} dt$ so that $h'(x) = (g(x))^{-2}$. Note that $h(x) > 0$ on $(0, \infty)$. Because f is bounded, there exists $B > 0$ such that $f(x) = g(x)h(x) \le B$ on $[0, \infty)$. Therefore $g^2(x)h^2(x) \le B^2$ and thus $h'(x)/h^2(x) \ge 1/B^2$ on $(0, \infty)$. Integrating this inequality yields

$$\frac{1}{h(1)} - \frac{1}{h(x)} = \int_1^x \frac{h'(t)}{h^2(t)}\, dt \ge \int_1^x \frac{1}{B^2}\, dt = \frac{1}{B^2}(x-1) \text{ on } [1, \infty).$$

Therefore

$$\frac{1}{B^2}(x-1) \le \frac{1}{h(x)} + \frac{1}{B^2}(x-1) \le \frac{1}{h(1)} \text{ on } [1, \infty),$$

which is a contradiction.

Editor's Note. By letting $B(x) = c\sqrt{x}$, the same argument shows that $f(x)/\sqrt{x}$ is also unbounded. On the other hand, the function $g(x) = \sqrt{x+2}$ shows that it is possible for $f(x)/(\sqrt{x+2}\ln(x+2))$ to be bounded on $[0, \infty)$.

REVIEWS

PAUL J. CAMPBELL, *Editor*
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Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Beardon, Alan F., *Creative Mathematics: A Gateway to Research*, Cambridge University Press, 2009; x + 110 pp, $27.99(P). ISBN 978-0-521-13059-2.

Problem books abound. Naturally, most focus on solving the problems. But there can be another, larger aim: expanding on the problems and furthering additional mathematical discovery. This book offers 11 problems, each with a solution and more problems; and then those further problems are discussed and generalizations urged. The book begins with a succinct eight pages on how to write mathematics and give a presentation, on the grounds that writing and communicating a careful solution to a problem itself stimulates further thought and new ideas. Some of the problems require linear algebra, others modular arithmetic, and a few some probability (mostly finite); one applies Taylor series with remainder to realize a limit. This is an inspiring book; I wish the price could be lower.

Alsina, Claudi, and Roger B. Nelsen, *When Less Is More: Visualizing Basic Inequalities*, MAA, 2009; xix + 190 pp, $59.95; member price $48.95. ISBN 978-0-88385-342-9.

Dip anywhere into this book and you will learn something new to you: Guha's inequality as a lemma to an easy proof of the inequality of the means, Simpson's paradox in statistics as an illustration of the mediant inequality, not one but three geometric proofs of the Cauchy-Schwarz inequality, and the use of majorization to prove inequalities. This book concentrates on geometric inequalities and indeed aims "to present a methodology for producing mathematical visualization of inequalities." Each of the nine chapters is devoted to a method, such as representing numbers geometrically, or using incircles, circumcircles, reflections, rotations, transformations, or graphs of functions. Each chapter ends with challenges to apply its method, and solutions are given to all challenges.

Hardy, Michael, and Catherine Woodgold, Prime simplicity, *Mathematical Intelligencer* 31 (4) (2009) 44–52.

Do you think that Euclid proved the existence of infinitely many prime numbers by contradiction? You may think so, and I bet that you know how to do it that way—but Euclid didn't do it that way, despite the fact that lots of your colleagues (including some famous ones) have written that he did. Euclid in fact gave a constructive proof, that "there are more prime numbers than any proposed multitude of prime numbers"—not that there are an infinite number of them, since the concept of an actual (as opposed to potential) infinity was not part of Greek thought. This study of the history of Euclid's proof, with 147 references, is remarkably thorough. The authors conclude, however: "When and how did the error become the prevailing doctrine? We have no answer." Though the authors find no single infection as the source, this virus surely is a consequence of the modern curriculum's abandonment of the custom in the nineteenth century (and earlier) of direct study of Euclid's work itself.

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Stein, James D., *How Math Can Save Your Life (and Make You Rich, Help You Find the One, and Avert Catastrophes)*, Wiley, 2010; xiv + 242 pp, $24.95. ISBN 978-0-47043-775-9.

Well, if there was ever a title to sell a book about math, this should be it! In the table of contents, each chapter title is paired with three intriguing questions (e.g., "Will refinancing your house actually save money?"). The writing is informal and brisk, and author Stein is a Berkeley Ph.D. in mathematics with a long career in university teaching. He considers mathematical aspects of all kinds of everyday topics: service contracts for appliances, strategy in football, finding a mate, picking lottery numbers, risky surgery, hybrid cars, financial indexes, teaching children arithmetic, and damage from disasters. The main tool is expected value, with contributions from symbolic logic, game theory, and regression to the mean. I was put off a bit by anti-algebra editorializing in the Introduction: "outside of the people [in the sciences, engineering, and investments] almost nobody needs algebra or ever uses it." But I was more troubled by bad arithmetic and the wrong conclusion about auto insurance policies (p. 14), no mention of utility in regard to expected value, and the otherwise-ingenious "Tulip Indexes" that unfortunately divide the S&P stock index and mean new-home prices in *current* dollars by mean household income in *constant* (inflation-adjusted) dollars without observing that fact.

Bayley, Melanie, Alice's adventures in algebra: Wonderland solved, *New Scientist* issue 2739 (16 December 2009). Algebra in Wonderland, *New York Times* (7 March 2010), `http://www.nytimes.com/2010/03/07/opinion/07bayley.html`.

Pycior, Helena M., At the intersection of mathematics and humor: Lewis Carroll's *Alices* and symbolical algebra, *Victorian Studies* 28 (1) (Autumn 1984) 149–170.

Devlin, Keith, The hidden math behind *Alice in Wonderland*, `http://www.maa.org/devlin/devlin_03_10.html` .

Wilson, Robin J., *Lewis Carroll in Numberland: His Fantastical Mathematical Logical Life: An Agony in Eight Fits*, W.W. Norton, 2008; xi + 237 pp., $24.95. ISBN 978-0-393-06027-0.

The recent release in March of the new film "Alice in Wonderland" will no doubt regenerate interest in Lewis Carroll (Charles Dodgson) and his works. Mathematics instructors may be able to use that renewed interest as a "teachable moment," thanks to author Bayley, a doctoral candidate in Victorian literature. She claims that in the *Alice* book Dodgson satirized and argued against the absurdity of the "new mathematics" of his day, which she takes to be imaginary numbers, symbolical algebra, projective geometry, and quaternions. Commentator Devlin summarizes Bayley's arguments sympathetically, even agreeing with the highly-implausible assertion that without the "mathematical undercurrents" the book would never have achieved stardom! Earlier, author Pycior looked into Dodgson's struggle against symbolical algebra and his fusion of mathematics with humor. Did Dodgson's mathematical colleagues react to *Alice*? Neither Robin Wilson nor Martin Gardner, author of *The Annotated Alice*, has weighed in yet on this latest cluster of claims.

Denning, Peter J., and Peter A. Freeman, Computing's paradigm, *Communications of the Association for Computing Machinery* 52 (12) (December 2009) 28–30.

A recent curriculum proposal at my college would reclassify academic departments as arts, humanities, social sciences, or natural and physical sciences. Curiously, both mathematics and computer science, currently classified as ("unnatural") sciences, were left out (what's the message in that?). A colleague in my department and I asserted that they are—"of course"—humanities. There is confusion in the public mind about mathematics and computer science; the prevailing erroneous view (even among college faculty) is that they both deal primarily with numbers. Authors Denning and Freeman ask what characterizes *computing* and present for it a paradigm, that is, "a belief system and its associated practices, defining how a field sees the world and approaches the solutions of problems." Their main concern is "reconciling the engineering and science views of computing," and they accept that "computing is a fourth great domain of science alongside the physical, life, and social sciences." From the three sub-paradigms of mathematics, science, and engineering, they synthesize the computing paradigm as focusing on "information processes—natural or constructed... discrete or continuous." There is no mention of numbers whatever.

NEWS AND LETTERS

50th International Mathematical Olympiad

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Problems

1. Let n be a positive integer and let $a_1, \dots, a_k$ $(k \geq 2)$ be distinct integers in the set $\{1, \dots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, \dots, k-1$. Prove that n does not divide $a_k(a_1 - 1)$.
2. Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L, and M be the midpoints of the segments BP, CQ, and PQ respectively and let Γ be the circle passing through K, L, and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.
3. Suppose that $s_1, s_2, s_3, \dots$ is a strictly increasing sequence of positive integers such that the subsequences $s_{s_1}, s_{s_2}, s_{s_3}, \dots$ and $s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \dots$ are both arithmetic progressions. Prove that $s_1, s_2, s_3, \dots$ is itself an arithmetic progression.
4. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45°$. Find all possible values of $\angle CAB$.
5. Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a,\ f(b),\ \text{and}\ f(b + f(a) - 1).$$

   (A triangle is *non-degenerate* if its vertices are not collinear.)
6. Let $a_1, a_2, \dots, a_n$ be distinct positive integers and let M be a set of $n-1$ integers not containing $s = a_1 + a_2 + \dots a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \dots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any points in M.

Solutions Following are the essential ideas for each problem. These solution sketches are adapted from [**1**] and details and alternatives are in the forum.

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1. Prove inductively that $n \mid a_1(a_i - 1)$ for $i = 2, \ldots, k$. The case $i = 2$ is a hypothesis so assume true for $i > 2$. Then $n \mid a_1(a_i - 1)$ and $n \mid a_i(a_{i+1} - 1)$, so $n \mid (a_1 a_i - a_1)a_{i+1} - a_1 a_i + a_1$ and $n \mid a_1 a_i a_{i+1} - a_1 a_i$. Subtracting the first from the second, we obtain $n \mid a_1 a_{i+1} - a_1$ so the induction is complete. Now $n \mid a_1 a_k - a_1$ and if $n \mid a_k a_1 - a_k$, then $n \mid a_1 - a_k$ which is impossible.

   This problem was proposed by Ross Atkins of Australia.

2. The circle Γ is tangent to line PQ if and only $\angle MLK = \angle QMK$. Since MK is parallel to AB, it follows that $\angle AQP = \angle MLK$. Since MK and ML are mid-lines in $\triangle PQB$ and $\triangle PQC$ respectively, it follows that $\angle PAQ = \angle KML$. Therefore $\triangle APQ \sim \triangle MKL$. Then $AP/AQ = MK/ML = BQ/PC$ and so $AP \cdot PC = AQ \cdot BQ$. But $AP \cdot PC$ is the power of P with respect to the circle with center O. Then $AP \cdot PC = R^2 - OP^2$. Similarly $AQ \cdot BQ = R^2 - OQ^2$ and so $OP = OQ$.

   This problem was proposed by Sergei Berlov of Russia.

3. Suppose $s_{s_i} = A + i \cdot d_A$ and $s_{s_i+1} = B + i \cdot d_B$ where $A, B, d_A, d_B > 0$. Since s_i is an increasing sequence $s_{s_i+1} > s_{s_i}$. Note that $s_{s_i+1} - s_{s_i} = (B - A) + i(d_B - d_A)$ and $s_{s_{i+1}} - s_{s_i+1} = (A + d_A - B) + i(d_A - d_B)$ are arithmetic progressions and their common differences add to zero. If the first of the common differences $d_B - d_A$ is strictly positive, then the other common difference $d_A - d_B$ must be strictly negative, and so eventually $s_{s_{i+1}} - s_{s_i+1}$ must be negative, a contradiction to being increasing. Likewise, if $d_A - d_B$ is strictly positive, then eventually $s_{s_i+1} - s_{s_i}$ must be negative, also a contradiction. Hence $d_A = d_B$ and $s_{s_1}, s_{s_2}, \ldots$ and $s_{s_1+1}, s_{s_2+1}, \ldots$ have the same common difference, say d. Establish that $s_i \geq i$ by induction. Then $s_{s_{i+1}} - s_{s_i} \geq s_{i+1} - s_i \geq 0$. Since $d = s_{s_{i+1}} - s_{s_i}$, we see $s_{i+1} - s_i$ is bounded. The difference achieves a maximum $s_{a+1} - s_a = M$ and minimum $s_{b+1} - s_b = m$. Let $s_a = k$. Let $s_b = l$.

   Then $s_{s_{s_{a+1}}} - s_{s_{s_a}} = s_{s_{s_a+M}} - s_{s_{s_a}} = s_{s_{k+M}} - s_{s_k} = M \cdot d$ since s_{s_i} is an arithmetic progression with common difference d. Since M is the maximum of $s_{i+1} - s_i$, and the average value of $s_{i+1} - s_i$ from $s_{s_{s_a}}$ to $s_{s_{s_{a+1}}}$ is M, it follows $s_{s_{s_a}+1} - s_{s_{s_a}} = M$. But $s_{s_i+1} - s_{s_i}$ is constant, so it equals M. By a similar argument using that m is the minimum of $s_{i+1} - s_i$, we have $s_{s_i+1} - s = m$. Hence $M = m$ and the given sequence is arithmetic.

   This problem was proposed by Gabriel Carroll of the USA.

4. Let $\alpha = \angle DAC$. Then $\angle CAB = 2\alpha$, $\angle BCA = \angle CBA = 90° - \alpha$, and $\angle EBC = 45° - \alpha/2$. Consider $\triangle EKC$ with $\angle KCE = 45° - \alpha/2$, $\angle CEK = 3\alpha/2$ and $\angle CKE = 135° - \alpha$. Finally, $\angle CKA = 135°$. From elementary trigonometry $CB = 2CD = 2AC\sin(\alpha)$.

   Applying the Law of Sines to $\triangle BEC$ and substituting for CB

$$EC = 2AC\sin(\alpha)\frac{\sin(45° - \alpha/2)}{\sin(45° + 3\alpha/2)}. \tag{1}$$

   Apply the Law of Sines to $\triangle AKC$ and simplify to obtain $KC = \sqrt{2}AC\sin(\alpha/2)$. Finally, apply the Law of Sines to $\triangle EKC$ and rearrange to obtain $EC = KC\sin(135° - \alpha)/\sin(\alpha/2)$. Combining

$$EC = \sqrt{2}AC\sin(\alpha/2)\frac{\sin(135° - \alpha)}{\sin(3\alpha/2)}. \tag{2}$$

   Then equating (1) and (2) and cancelling AC

$$2\sin(\alpha)\frac{\sin(45° - \alpha/2)}{\sin(45° + 3\alpha/2)} = \sqrt{2}\sin(\alpha/2)\frac{\sin(135° - \alpha)}{\sin(3\alpha/2)}. \tag{3}$$

   Solving this equation, $\alpha = 30°$ or $\alpha = 45°$, so $\angle CAB = 60°$ or $90°$.

   The problem was suggested by Jan Vonk, Belgium, Peter Vandendriessche, Belgium and Hojoo Lee, Korea.

5. First note that if a triangle has positive integer side lengths 1, a, b, then by the triangle inequality $a - 1 \leq b \leq a + 1$. If the triangle is non-degenerate, then $a = b$. Using $a = 1$, then $f(b) = f(b + f(a) - 1)$. Now the claim is that $f(a) = 1$, since otherwise if $f(a) > 1$ then f is periodic of period $f(a) - 1$, and f is bounded above. Then choosing a larger than twice the upper bound violates the triangle inequality. Using $b = 1$, then a, $f(1) = 1$, $f(f(a))$ are the side lengths of a triangle, so $a = f(f(a))$ for all a. Thus f is injective.

   Now assume $f(2) = k > 2$. Hence $f(b) - 1 \leq f(b + f(2) - 1) \leq f(b) + 1$. Then check the 3 possibilities for $f(b + f(2) - 1)$:

   (a) $f(b + f(2) - 1) = f(b)$, so $f(2) = 1$ which is impossible.

   (b) $f(b + f(2) - 1) = f(b) - 1$, so set $k = f(2) - 1$, so $f(b + k) = f(b) - 1$. By induction $f(b + n \cdot k) = f(b) - n$. Choosing $n = f(b) - 1$ leads to function value 1, contradicting injectivity.

   (c) $f(b + f(2) - 1) = f(b) + 1$. Set $b = 1$, $f(2) - 1 = k$, so $f(1 + k) = f(1) + 1$. Inducting, $f(1 + n \cdot k) = n + 1$. Now if $k > 1$, then $1 \leq k - 1 < k + 1 < 1 + n \cdot k$. This means that $f(k - 1) = 1 = f(1)$ or $k = 2$ implies $f(2) = 3$ and $f(b + 2) = f(b) + 1$ and finally $f(2) = 3$ and $f(5) = 3$ which is impossible. Conclude that $k = 1$, so $f(b + 1) = f(b) + 1$ and $f(n) = n$.

   This problem was proposed by Bruno Le Floch of France.

6. Induct on n. The cases $n = 1$ and $n = 2$ are easy. For $n \geq 3$, without loss of generality let a_n be the largest jump size and let m_1 be the smallest element of M. Consider 3 cases.

   (a) If $m_1 < a_n$ and $a_n \notin M$, then begin with a jump of size a_n. That jump avoids m_1, and the induction hypothesis means that the grasshopper can arrange the remaining $n - 1$ jumps to avoid the remaining $n - 2$ values of M.

   (b) If $m_1 < a_n$ but $a_n \in M$, say $a_n = m_j$ for some j, then consider the starting two-jump sequences $(a_1, a_n), \ldots (a_{n-1}, a_n)$. There are $n - 1$ of these sequences, and the landing values are all distinct and different from m_j. Therefore there are not enough forbidden values in M to block all of them. For some i, the grasshopper can start with two safe jumps of size a_i and a_n. These two jumps take the grasshopper past m_1 and m_j, and by induction the grasshopper can arrange the remaining $n - 2$ jumps to avoid the remaining $n - 3$ values of M.

   (c) If $m_1 \geq a_n$ the grasshopper needs a different strategy. Begin with jump a_n, ignore the value m_1, and arrange the remaining jumps to avoid the remaining $n - 2$ values of M other than m_1. If this arrangement avoids m_1, the proof is done. Otherwise, suppose that the grasshopper lands on m_1 just before making a jump of size a_i. Then modify the jump sequence by exchanging jumps a_n and a_i. Then verify that the modified sequence avoids all the values of M.

   This solution is by Anton Mellit, IMO observer with the Ukraine delegation, and Ilya Bogdanov, IMO observer with the Russian delegation with simplifications by Brian Basham, a mathematics student at MIT.

   Immediately following the IMO, Terry Tao hosted a collaborative solution on his blog site as a "mini-polymath project," [**3**]. The "polymath" collaborative solution continued two days [**4**] until the contributors agreed upon a solution. Terry Tao followed with an analysis of the polymath process, [**5**]. Michael Nielsen wrote up 5 variant proofs from the collaboration [**2**].

   This problem was proposed by Dmitry Khramtsov of Russia.

2009 International Mathematical Olympiad Results At the IMO 530 young mathematicians from 104 countries competed on July 15–16, 2009. The USA team ranked 6th among all 104 participating countries. The USA team has consistently finished in the top ten at the IMO. As part of the 50th anniversary of the IMO, Terry Tao and 5 other famous mathematicians who were IMO medalists gave commemorative lectures. The students visited a mag-lev train demonstration project, the North Sea resort island Wangerooge, and the historic Bremen city center.

- John Berman, a graduate of John T. Hoggard High School, Wilmington, NC, won a Gold medal.
- Wenyu Cao, a student at Phillips Academy, Andover, Massachusetts won a Silver medal.
- Eric Larson, who graduated from South Eugene High School, Eugene, OR won a Gold medal.
- Delong Meng who graduated from Baton Rouge Magnet School, Baton Rouge, LA won a Silver medal.
- Evan O'Dorney who attends the Venture School and is from Danville CA, won a Silver medal.
- Qinxuan Pan, who graduated from Wooton High School in Rockville MD, won a Silver medal.

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